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Constructive reverse investigations into differential equations

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Abstract: We study Picard's Theorem and Peano's Theorem from a constructive reverse perspective. This means that we have to change our focus from global properties to local properties. We also extend the theory of *pointwise* continuously differentiable functions to include Rolle's Theorem, the Mean Value Theorem, and the full Fundamental Theorem of Calculus.

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1 Introduction

Under the program of constructive reverse mathematics, many theorems have been proven equivalent over Bishop's constructive mathematics (BISH) to the Uniform Continuity Theorem [12, 14]:

> **UCT** Every pointwise continuous mapping of a compact[1] metric space into a metric space is uniformly continuous.

By building **UCT** into his definition of continuity, Bishop elegantly circumvented the decision of whether to accept it as a principle or not. In his own words he deemed "the concept of a [pointwise] continuous function [...] not relevant" [9, p. 3]. In the same fashion, Bishop focused on functions that are differentiable in a uniform way, and was not interested in pointwise differentiability. We believe that the contrast of pointwise versus uniform properties for continuity and differentiability is interesting.

In the tradition of Bishop we make free use of the axiom of countable and dependent choice. We will, however, explicitly mention this in these occasions

[1] Following common practice in constructive mathematics, we take totally boundedness together with completeness as our definition of compactness.

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This paper has three goals. The first being to contribute to the program of constructive reverse mathematics. The second, related goal, is to highlight and understand the difference between local and uniform definitions of continuity and differentiability. The third goal is to, within BISH, prove theorems using pointwise definitions of differentiability; thus continuing work begun in [18]. Although we are thus working in Bishop-style informal mathematics, we believe that this research could be carried out in a suitable formal framework like IZF [15, 4], CZF [3], or HA^{ω} [25, 26].

Section 2 studies two varieties of **UCT**. They both turn out to be equivalent and will play a role in the rest of the paper. In Section 3 we investigate pointwise differentiability. A constructive proof of Rolle's Theorem without additional assumptions, commonly made in the constructive literature, is presented.

In Section 4 and 5 we prove that versions of Picard's Theorem and Peano's Theorem are equivalent to **UCT**. As far as we know, these are the first results on differential equations in constructive reverse mathematics. For some other results on constructive existence of solutions of differential equations see e.g. [16], which deals amongst other things with Euler's method.

Being equivalent to **UCT**, there is no hope to prove these theorems in the framework of Russian Recursive Mathematics. In the last section we will strengthen this result and produce strong counterexamples. That means we will actually give an example of a recursive function for which these theorems fail to hold.

2 Uniform Continuity Theorems

Since there are many different notions of continuity commonly in use, we will specify the definitions we have in mind when talking about continuity throughout this paper.

Definition 1 *Consider two metric spaces (X, d_X) and (Y, d_Y). A function $f : X \to Y$ is called* **continuous**, *if for all $x \in X$ and for all $\varepsilon > 0$ there exists $\delta > 0$ such that for all $x' \in X$*

$$d_X\left(x', x\right) < \delta \implies d_Y\left(f(x), f(x')\right) < \varepsilon.$$

Furthermore, it is called **uniformly continuous**, *if δ does not depend on x.*

First, we prove an extension result for pointwise continuous functions that is needed later, but is also of interest by itself.

Lemma 2 *Consider an arbitrary metric space* (X, d_X) *and a complete metric space* (Y, d_y). *Furthermore assume that* $D \subset X$ *is a dense subset and* $f : D \to Y$ *a function such that for every* $x \in X$ *and* $\varepsilon > 0$ *there exists* $\delta > 0$ *with*

$$\forall y, z \in D\left((d_X(x,y) < \delta \wedge d_X(x,z) < \delta) \implies d_Y(f(y), f(z)) < \varepsilon\right). \tag{1}$$

Then there exists a unique continuous function $\tilde{f} : X \to Y$ *such that* $\tilde{f}(x) = f(x)$ *for all* $x \in D$.

Proof Since D is dense, for every $x \in X$ we can find a sequence $(x_n)_{n\geqslant 1}$ in D that converges to x. Property (1) now ensures that $(f(x_n))_{n\geqslant 1}$ is Cauchy and hence converges. Furthermore, the limit is not dependent on the choice of the sequence $(x_n)_{n\geqslant 1}$, and thus it makes sense to denote this limit by $\tilde{f}(x)$.[2] Using unique choice we get a function $\tilde{f}$, which is continuous: for let $x \in X$ and $\varepsilon > 0$ be arbitrary. Choose $\delta > 0$ such that (1) is satisfied. Now consider $y \in X$ such that $d_X(x, y) < \delta$. By the construction of $\tilde{f}$ we can find $x' \in D$ and $y' \in D$ with $d(x', x) < \delta$ and $d(y', y) < \delta$. By (1) we therefore have $d_Y(\tilde{f}(x), f(x')) < \varepsilon$, $d_Y(\tilde{f}(y), f(y')) < \varepsilon$ and $d(f(x'), f(y')) < \varepsilon$. It follows that

$$d_Y(\tilde{f}(x), \tilde{f}(y)) \leqslant d_Y(\tilde{f}(x), f(x')) + d_Y(f(x'), f(y')) + d_Y(f(y'), \tilde{f}(y)) \leqslant 3\varepsilon,$$

whence $\tilde{f}$ is continuous. Since for any $x \in D$ the constant sequence $(x)_{n\geqslant 1}$ converges to x, also $\tilde{f}(x) = f(x)$. To see that $\tilde{f}$ is unique, consider another continuous function $g : X \to Y$ such that $f(x) = g(x)$ for all $x \in D$. Now assume that $d(\tilde{f}(x_0), g(x_0)) > 0$ for some $x_0 \in X$. Then, because we are dealing with continuous functions, there exists a neighbourhood U of x_0 such that $d(\tilde{f}(x), g(x)) > 0$ for all $x \in U$. Since D is dense, the intersection $D \cap U$ is inhabited and we get a contradiction; so $\tilde{f} = g$. □

Working within BISH, we are interested in the following three principles, where a, b are real numbers with $a < b$:

UCT$_{[a,b]}$ Every continuous function $f : [a, b] \to \mathbb{R}$ is uniformly continuous.

[2]Here countable choice is used in choosing a sequence $(x_n)_{n\geqslant 1}$. This is, however, for convenience only. Countable choice is avoidable here, if one used Dedekind reals (more details can be found in [17]). One could then define

$$\tilde{f}(x) = \bigcap_{n\in\mathbb{N}} \left\{ y \in Y \mid \exists z \in X \left(|x - z| < \frac{1}{n} \wedge y < f(z) \right) \right\}.$$

Property (1) ensures that this set is order located, which is an additional requirement on a Dedekind real in the constructive treatment.

> $\mathbf{BUCT}_{[a,b]}$ Every bounded, continuous function $f : [a,b] \to \mathbb{R}$ is uniformly continuous.

> $\mathbf{LUCT}_{[a,b]}$ Every continuous function $f : [a,b] \to \mathbb{R}$ is locally uniformly continuous.

Where *locally uniformly continuous* is defined as follows:

Definition 3 *A continuous function $f : [a,b] \to \mathbb{R}$ is* **locally uniformly continuous**, *if for every $x \in [a,b]$ there exists $h > 0$ such that f is uniformly continuous on $[x-h, x+h] \cap [a,b]$.*

From [12] we know that $\mathbf{UCT}_{[0,1]}$ is equivalent to **UCT**. Since every continuous function is locally bounded, the following implications hold

$$\mathbf{UCT} \implies \mathbf{BUCT}_{[a,b]} \implies \mathbf{LUCT}_{[a,b]}.$$

We prove the reverse implications for functions defined on the unit interval. The general cases easily follow by scaling. In order to prove the next implication we first introduce the following principle:

> $\mathbf{AS}_{[0,1]}$ If $(x_n)_{n\geqslant 1}$ is a sequence of real numbers that is bounded away from every point in $[0,1]$ then $(x_n)_{n\geqslant 1}$ is eventually bounded away from the entire interval.

In [6], this principle has been shown to be equivalent to a version of Brouwer's Fan theorem, which itself is weaker than **UCT** [5], but stronger than the Fan theorem for decidable bars.[3] This means, in particular, that together with Corollary 3.4 in [13, Chapter 2] $\mathbf{AS}_{[0,1]}$ implies:

> **POS** Every positively valued, uniformly continuous function $f : [0,1] \to \mathbb{R}$ has a positive infimum.

We will show that $\mathbf{BUCT}_{[0,1]}$ is enough to show that $\mathbf{AS}_{[0,1]}$ holds.

Lemma 4 *If $(x_n)_{n\geqslant 1}$ is a sequence in $\mathbb{R}$ that is bounded away from every point in $[0,1]$, then there exists a subsequence $(x_{k_n})_{n\geqslant 1}$ such that for all $n \in \mathbb{N}$ one can decide whether*

$$x_{k_n} \in [0,1] \vee x_{k_n} \notin [0,1],$$

[3] It is an open question in constructive reverse mathematics, whether any of these implications are actually strict.

and there exist positive numbers $(\varepsilon_n)_{n\geqslant 1}$ such that for all $n,m \in \mathbb{N}$ with $m > n$

$$\big(x_{k_n} \in [0,1] \wedge x_{k_m} \in [0,1]\big) \implies |x_{k_n} - x_{k_m}| > \varepsilon_n. \tag{2}$$

Furthermore $(x_{k_n})_{n\geqslant 1}$ is bounded away from $[0,1]$ if and only if $(x_n)_{n\geqslant 1}$ is.

Proof Let $(x_n)_{n\geqslant 1}$ be a sequence in $\mathbb{R}$ that is bounded away from every point in $[0,1]$. Since $(x_n)_{n\geqslant 1}$ is bounded away from 0 and 1, there exists N such that for all $i \geqslant N$ we can decide

$$x_i \in [0,1] \vee x_i \notin [0,1].$$

Now, with the help of dependent choice, define a subsequence the following way: start by setting $k_1 = N$. Assume we have constructed k_n for some n. If $x_{k_n} \notin [0,1]$ let $k_{n+1} = k_n + 1$. If $x_{k_n} \in [0,1]$ there exists $\varepsilon_n > 0$ and k_{n+1} such that for all $i \geqslant k_{n+1}$

$$|x_{k_n} - x_i| > \varepsilon_n.$$

Clearly, the so defined subsequence satisfies (2). Now assume that there exists M such that

$$x_{k_i} \notin [0,1] \text{ for all } i \geqslant M. \tag{3}$$

Then there cannot be a $j \geqslant k_M$ with $x_j \in [0,1]$: for assume such a j exists. Then find

$$j' = \min\{i : k_M < i \leqslant j \wedge x_i \in [0,1]\}.$$

The construction therefore ensures that

$$x_{k_{M+(j'-k_M)}} = x_{j'} \in [0,1];$$

a contradiction to (3), and thus $x_j \notin [0,1]$ for all $j \geqslant k_M$. Since $(x_n)_{n\geqslant 1}$ is also bounded away from 0 and 1, the sequence is bounded away from the entire interval. □

Lemma 5 **BUCT**$_{[0,1]}$ *implies* **AS**$_{[0,1]}$ *(and therefore* **POS***).*

Proof Given a sequence $(x_n)_{n\geqslant 1}$ that is bounded away from every point in $[0,1]$, construct a subsequence and $(\varepsilon_n)_{n\geqslant 1}$ as in Lemma 4. Furthermore, we may, perforce, assume that ε_n is decreasing. Since $(x_n)_{n\geqslant 1}$ is bounded away from every point in $[0,1]$ we may also assume that if $x_{k_n} \in [0,1]$ then $0 < x_{k_n} - \varepsilon_n < x_{k_n} + \varepsilon_n < 1$. This ensures that for any given $x \in [0,1]$ at most one term of the sum

$$\sum_{n : x_{k_n} \in [0,1]} \max\left\{0, \left|1 - \frac{2(x_{k_n} - x)}{\varepsilon_n}\right|\right\}$$

is nonzero. The so defined function $f : [0,1] \to \mathbb{R}$ is easily seen to be well-defined and continuous, and, furthermore, satisfies $0 \leqslant f \leqslant 1$. We can therefore apply $\mathbf{BUCT}_{[0,1]}$ to ensure that f is uniformly continuous. So there exists $N \in \mathbb{N}$ such that for $x, y \in [0,1]$

$$|x - y| < 2^{-N} \implies |f(x) - f(y)| < \frac{1}{2}. \tag{4}$$

Now assume that there exists $n \geqslant N$ such that $x_n \in [0,1]$. Then $f(x_n) = 1$ and $f(x_n + \varepsilon_n/2) = 0$ a contradiction to (4). Hence $x_n \notin [0,1]$ for all $n \geqslant N$, and since $(x_n)_{n\geqslant 1}$ is also eventually bounded away from 0 and 1, it is eventually bounded away from the entire interval $[0,1]$. □

Proposition 6 $\mathbf{BUCT}_{[0,1]} \implies \mathbf{UCT}_{[a,b]}$

Proof Consider a continuous function $f : [a,b] \to \mathbb{R}$. With the work in [12], it suffices to show that f is bounded. Since f is continuous, so is the function $g : [a,b] \to \mathbb{R}$ defined by

$$g(x) = \frac{1}{\max\{1, |f(x)|\}}.$$

By virtue of the construction of g, the following inequalities hold:

$$\forall x \in [a,b](0 < g(x) \leqslant 1).$$

As we assume $\mathbf{BUCT}_{[0,1]}$, g is uniformly continuous. Also, using Lemma 5, we can find an $\varepsilon > 0$ such that

$$\forall x \in [0,1](\varepsilon < g(x) \leqslant 1).$$

So $|f|$ is bounded by $\max\{\varepsilon^{-1}, 1\}$. □

The more interesting implication is

Proposition 7 $\mathbf{LUCT}_{[0,1]} \implies \mathbf{BUCT}_{[0,1]}$

Proof Let $f : [0,1] \to \mathbb{R}$ bounded and continuous. Furthermore, without loss of generality, we assume that $f(0) = f(1) = 0$ and that $0 \leqslant f \leqslant 1$. Let I_n denote the open interval $(1 - \frac{1}{2^n}, 1 - \frac{1}{2^{n+1}})$. For any $x \in I_n$, we define $g(x)$ to be

$$\frac{1}{2^{n+1}} f(2^{n+1}x - 2(2^n - 1)).$$

Lemma 2 yields the existence of a continuous map $\tilde{g} : [0,1] \to \mathbb{R}$, such that $\tilde{g}(x) = g(x)$ for all $x \in I_n$ and, furthermore, $\tilde{g} \leqslant 1 - x$. Figure 1 is an illustration of the idea behind the construction of the function $\tilde{g}$. Now because we assume $\mathbf{LUCT}_{[0,1]}$, there exists N such that $\tilde{g}$ is uniformly continuous on $[1 - 2^{N-1}, 1]$. Since $f = h \circ \tilde{g} \circ h'$ for some linear, and therefore uniformly continuous, functions h, h', we conclude that f is uniformly continuous. □

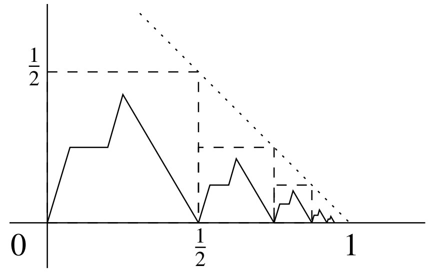

Figure 1: Construction of the function $\tilde{g}$.

We have now proved that $\mathbf{LUCT}_{[a,b]}$ is equivalent to **UCT**. We can easily generalise this result to arbitrary (compact) metric spaces and **LUCT** as follows.

Let (X,ρ) be a compact metric space and Y be a metric space.

Definition 8 *A continuous function $f : X \to Y$ is* **locally uniformly continuous**, *if for every $x \in X$ there exists $h > 0$ such that f is uniformly continuous on $\{y \in X | \rho(x,y) \leqslant h\} \cap X$.*

LUCT is the following principle:

> **LUCT** Every continuous function of a compact metric space into a metric space is locally uniformly continuous.

The following implications can now be seen to hold:

$$\mathbf{UCT} \implies \mathbf{LUCT} \implies \mathbf{LUCT}_{[0,1]} \implies \mathbf{UCT}.$$

So **UCT** and **LUCT** are equivalent. We will use this in the Sections 4 and 5.

3 Differentiation

Just like we do not restrict our view to functions that are uniformly continuous (on compacts), we will not presuppose that every differentiable function on a compact interval is uniformly differentiable either.

Definition 9 *Let f be a continuous function on $[0,1]$. We say that f is* **differentiable** *if there exists a continuous function g on $[0,1]$ such that for each x in $[0,1]$ and $\varepsilon > 0$, there exists $\delta > 0$ such that if y in $[0,1]$ and $|x-y| < \delta$, then*

$$|f(y) - f(x) - g(x)(y-x)| \leqslant \varepsilon |y-x|.$$

The function g is called the **derivative** *of f.*

If f is a differentiable function, we will often write its derivative as f'. Note that every function has at most one derivative.

To contrast this version of differentiability with the two uniform ones that we will see later, we will also call it **continuous differentiability** to emphasise that the derivative is (pointwise) continuous, or **pointwise differentiability**, to stress that δ depends on x.

Rolle's theorem is vital for the development of Analysis. The classical version states that if $f : [a,b] \to \mathbb{R}$ is continuously differentiable and $f(a) = f(b) = 0$, then there exists a point $\xi \in [a,b]$ such that $f'(\xi) = 0$. It is not surprising that one cannot hope to find a constructive proof of this theorem. In fact, a Brouwerian counterexample can be found in [24]. Nevertheless, there is hope to prove the following approximate version.

Theorem 10 *If $f : [a,b] \to \mathbb{R}$ is continuously differentiable and $f(a) = f(b) = 0$, then for every $\varepsilon > 0$ there exists $x \in [a,b]$ with $|f'(x)| < \varepsilon$.*

Unfortunately the proof in [9] assumes that the function is differentiable in a uniform way. Recursive proofs, such as the one found in [2], make use of an unbounded search to find a point that satisfies the conclusion. Using dependent choice we can give a proof without any of these additional assumptions. To our knowledge this is the first such proof. First, though, we need to establish some lemmas.

Lemma 11 *If $f : [a,b] \to \mathbb{R}$ is continuously differentiable such that $f'(x) > \varepsilon$ for all $x \in [a,b]$ then it is impossible that $f(a) > 0$ and $f(b) < 0$.*

Proof Aberth's proof applies [2, Theorem 8.1]. □

Lemma 12 *If $f : [a,b] \to \mathbb{R}$ is continuously differentiable and $x \in [a,b]$ such that $|f'(x)| > 0$, then for each $\delta > 0$ there exists $y \in [a,b]$ such that $|x - y| < \delta$ and $|f(x) - f(y)| > 0$.*

Proof We first look at the case that $f'(x) > 0$. Let $\delta > 0$. By the continuity of f', we can find $\delta' > 0$ such that if $|z - x| \leqslant \delta'$, then $f'(z) > 0$. Take

$$y := x + \min\{\delta, \delta'\}.$$

We now apply Corollary 3 of [18] on the interval $[x, y]$. This gives us that $f(y) > f(x)$. The proof of the case that $f'(x) < 0$ is analogous and thus omitted. □

Corollary 13 *If $f : [a,b] \to \mathbb{R}$ is continuously differentiable and $x \in [a,b]$ such that $f'(x) > 0$, then for each $\delta > 0$ there exists $y \in [a,b]$ such that $|x - y| < \delta$ and $f(y) > f(x)$.*

Proof Follows from Lemmas 11 and 12. □

We now are in a position to prove Theorem 10.

Proof Let $\varepsilon > 0$. We may assume that $|f'(a)| > \varepsilon/2$ and $|f'(b)| > \varepsilon/2$, since otherwise we are done. In the cases that $f'(a) > \varepsilon/2$ and $f'(b) < \varepsilon/2$ or $f'(a) < \varepsilon/2$ and $f'(b) > \varepsilon/2$ we can apply an approximate version of the intermediate value theorem [13] to the continuous function f' to find an $x \in [a,b]$ such that $|f'(x)| < \varepsilon$. So let us, without loosing generality, assume that both $f'(a) > \varepsilon/2$ and $f'(b) > \varepsilon/2$.

The idea of the rest of the proof is to use a suitably modified interval halving procedure to obtain two sequences $(a_n)_{n\geqslant 1}$ and $(b_n)_{n\geqslant 1}$ and at the same time a binary sequence $(\lambda_n)_{n\geqslant 1}$, which keeps track whether a point with the desired property is found. If this happens the sequence $(\lambda_n)_{n\geqslant 1}$ becomes 1 from then on and the sequences $(a_n)_{n\geqslant 1}$ and $(b_n)_{n\geqslant 1}$ stabilise on this point. Of course we know that it is impossible that this never happens. Working without the assumption of Markov's principle though we have to, at least implicitly, produce a bound for this event. This is achieved, by choosing $(a_n)_{n\geqslant 1}$ and $(b_n)_{n\geqslant 1}$ to converge to a point y, which either has the desired property anyway or for which $f'(y)$ and continuity around this point contains enough information to find this bound.

Using dependent choice, we define a binary sequence $(\lambda_n)_{n\geqslant 1}$ and two sequences of real numbers $(a_n)_{n\geqslant 1}$ and $(b_n)_{n\geqslant 1}$ such that for every $n \in \mathbb{N}$

(1) $a_n \leqslant a_{n+1} \leqslant b_{n+1} \leqslant b_n$;

(2) $|b_n - a_n| \leqslant \left(\frac{2}{3}\right)^n |b-a|$;

(3) $\lambda_n = 0$ implies that $f(a_n) > 0 > f(b_n), f'(a_n) > \varepsilon/2$, $f'(b_n) > \varepsilon/2$ and $a_n < b_n$;

(4) $\lambda_n = 1$ implies that there exists $x \in [a,b]$ such that $|f'(x)| < \varepsilon$.

Notice that, since $f'(a) > 0$, it follows from Corollary 13 that there exists $a_0 \in [a,b]$ such that $f(a_0) > 0$. Similarly there exists $b_0 \in [a,b]$ with $f(b_0) < 0$. Again we might assume that $f'(a_0) > \varepsilon/2$ and $f'(b_0) > \varepsilon/2$, since otherwise we are done. Also set $\lambda_0 = 0$.

Now assume we have constructed λ_n, a_n and b_n for some $n > 0$. If $\lambda_n = 1$ simply set $\lambda_{n+1} = 1$, $a_{n+1} = b_{n+1} = a_n$. If $\lambda_n = 0$ consider $\xi = (a_n + b_n)/2$. Either $|f'(\xi)| < \varepsilon$, $f'(\xi) < \varepsilon/2$ or $f'(\xi) > \varepsilon/2$. In the second case we can use an approximate version of the intermediate value theorem to find $x \in [a,b]$ with $|f'(x)| < \varepsilon$. So in the first two cases set $\lambda_n = 1$, $a_{n+1} = b_{n+1} = a_n$. In the third case we can use Lemma 12 to find a point ξ'

such that $|\xi - \xi'| < \frac{1}{6}|b_n - a_n|$ and $f(\xi) \neq f(\xi')$. Now either $|f(\xi)| > 0$ or $|f(\xi')| > 0$. We will only deal with the first possibility, since the second possibility can be dealt with in an almost identical fashion. Once more we may assume that $f'(\xi) > \varepsilon/2$, because the other possibilities are obvious. If $f(\xi) > 0$ set $\lambda_{n+1} = 0$, $a_{n+1} = \xi$ and $b_{n+1} = b_n$. If $f(\xi) < 0$ set $\lambda_{n+1} = 0$, $a_{n+1} = a_n$ and $b_{n+1} = \xi$. Properties (1) and (2) ensure that the so defined sequences $(a_n)_{n\geqslant 1}$ and $(b_n)_{n\geqslant 1}$ are Cauchy, and converge to the same limit $y \in [a, b]$. For the final time, we may assume that $f'(y) > \varepsilon/2$, since we are done in the other cases. Since f' is continuous we can find $\delta > 0$ such that $f'(z) > \varepsilon/3$ for all $z \in [a, b]$ with $|z - y| < \delta$. Choose N such that $[a_N, b_N] \subset B_y(\delta)$. Now $\lambda_N = 0$ leads to a contradiction to Lemma 11 and hence $\lambda_N = 1$ and we are done. □

Corollary 14 *(Mean Value Theorem) If $f : [a, b] \to \mathbb{R}$ is continuously differentiable and $a < b$, then for every $\varepsilon > 0$ there exists $x \in [a, b]$ with*

$$\left| f'(x) - \frac{f(b) - f(a)}{b - a} \right| < \varepsilon.$$

Proof Apply Rolle's theorem 10 to

$$g(x) = f(x) - f(a) - \frac{f(b) - f(a)}{b - a}(x - a).$$

□

The next notion that we introduce, **uniform differentiability**, coincides with Bishop's notion of differentiability in [9], if we would suppose that the functions involved are uniformly continuous.

Definition 15 *Let f be a continuous function on $[0, 1]$. We say that f is* **uniformly differentiable** *if there exists a continuous function g on $[0, 1]$ such that for each $\varepsilon > 0$ there exists $\delta > 0$ such that if x, y in $[0, 1]$ and $|x - y| < \delta$, then*

$$|f(y) - f(x) - g(x)(y - x)| \leqslant \varepsilon |y - x|.$$

However, a definition in the spirit of Bishop is not the only thinkable restriction of Definition 9 to some kind of uniformity.

Definition 16 *Let f be a continuous function on $[0, 1]$. We say that f is* **uniformly continuously differentiable** *if the function f is differentiable and its derivative is uniformly continuous.*

It is not difficult to see that uniformly continuous differentiability follows from uniform differentiability. See also Proposition 2.2 in Chapter 6 of [25]. In fact both notions are equivalent.

Theorem 17 *Every real-valued function on* $[0,1]$ *is uniformly differentiable if and only if it is uniformly continuously differentiable.*

Proof Let $f:[0,1]\to\mathbb{R}$ be a uniformly differentiable function. Let $\varepsilon>0$. Determine $\delta>0$ such that for all $x,y\in[0,1]$, if $|x-y|<\delta$, then

$$|f(y)-f(x)-f'(x)(y-x)|<\frac{1}{2}\varepsilon|y-x|.$$

Let $x,y\in[0,1]$ such that $|x-y|<\delta$. Suppose now that $|x-y|>0$; we see that:

$$\begin{aligned}(f'(x)-f'(y))(x-y)&=-f'(x)(y-x)-f'(y)(x-y)\\&=(f(y)-f(x)-f'(x)(y-x))+\\&\qquad(f(x)-f(y)-f'(y)(x-y))\\&\leqslant|f(y)-f(x)-f'(x)(y-x)|+\\&\qquad|f(x)-f(y)-f'(y)(x-y)|\\&\leqslant 2\cdot\frac{1}{2}\cdot\varepsilon|y-x|=\varepsilon|y-x|.\end{aligned}$$

Similarly

$$(f'(x)-f'(y))(y-x)\leqslant\varepsilon|y-x|.$$

Hence $|f'(x)-f'(y)||y-x|\leqslant\varepsilon|y-x|$. So for all $n\in\mathbb{N}^+$ we have:

$$\frac{|f'(x)-f'(y)||y-x|}{|y-x|+n^{-1}}\leqslant\frac{\varepsilon|y-x|}{|y-x|+n^{-1}}.$$

By taking the limit $(n\to\infty)$ we conclude that $|f'(x)-f'(y)|<\varepsilon$. This shows that a uniformly differentiable function is uniformly continuously differentiable.

Conversely assume that $f:[0,1]\to\mathbb{R}$ is uniformly continuously differentiable, and let $\varepsilon>0$ be arbitrary. Since f' is uniformly continuous there exists $\delta>0$ such that for all $x,\xi\in[0,1]$

$$|x-\xi|<\delta\implies\left|f'(x)-f'(\xi)\right|<\frac{\varepsilon}{2}.$$

Now assume that $x,y\in[0,1]$ are such that $|y-x|<\delta$. Assume that $|x-y|>0$. Then by Corollary 14, there exists ξ such that $|x-\xi|<|x-y|<\delta$ and

$$\left|f'(\xi)-\frac{f(y)-f(x)}{y-x}\right|<\frac{\varepsilon}{2}.$$

Thus

$$\left| f'(x) - \frac{f(y) - f(x)}{y - x} \right| < \varepsilon.$$

Multiplying the last equation with $|y - x|$ gives

$$\left| f(y) - f(x) - f'(x)(y - x) \right| \leqslant \varepsilon |y - x|. \tag{5}$$

Notice that the function

$$g(x, y) := \left| f(y) - f(x) - f'(x)(y - x) \right| - \varepsilon |y - x|$$

is continuous and that $g(x, y) \leqslant 0$ on a dense subset of

$$\{(x, y) \in [0, 1]^2 : |x - y| \leqslant \delta\}.$$

We can therefore conclude that $g(x, y) \leqslant 0$ for all x, y with $|x - y| \leqslant \delta$. Thus Equation 5 holds for all such x, y and we are done. □

Analogous to the *Uniform Continuity Theorem*, we identify the **Uniform Differentiation Theorem** as follows:

> **UDT** Every differentiable function on the interval $[0, 1]$ is uniformly continuously differentiable.

Trivially **UCT** implies **UDT** over *BISH*. We can prove the following partial converse:

Proposition 18 **UDT** *implies* $\mathbf{AS}_{[0,1]}$.

Proof Similar to the proof of Lemma 5, we use spike functions—with the difference that we have to take differentiable spikes. Let $s_{x,\varepsilon} : [0, 1] \to [0, 1]$ be a differentiable spike with the following properties:

(1) $s_{x,\varepsilon}$ is uniformly continuously differentiable.

(2) $s_{x,\varepsilon}(x) = 1$,

(3) $s_{x,\varepsilon}(y) = 0$ for any y such that $|y - x| > \varepsilon/2$,

An example for such a family of functions would be defined by

$$s_{x,\varepsilon}(t) = \begin{cases} \frac{1}{2}\left(\cos\left(2\pi\varepsilon^{-1}(t - x)\right) + 1\right), & \text{if } t \in [x - \frac{\varepsilon}{2}, x + \frac{\varepsilon}{2}]; \\ 0, & \text{if } t \notin [x - \frac{\varepsilon}{2}, x + \frac{\varepsilon}{2}]; \end{cases}$$

For brevity's sake we will omit the proof that these particular functions are well-defined and satisfy the properties above (see also Lemma 2). Using the approximate version of the mean value theorem (Corollary 14) we can, for every x and ε, find a point $y \in [0,1]$ with

$$s'_{x,\varepsilon}(y) \geqslant \frac{1}{\varepsilon}. \tag{6}$$

Now consider a sequence $(x_n)_{n\geqslant 1}$ that is bounded away from every point in $[0,1]$. Again, let $(x_{k_n})_{n\geqslant 1}$ and $(\varepsilon_n)_{n\geqslant 1}$ be a sequences as in Lemma 4. Furthermore, we may, perforce, assume that $(\varepsilon_n)_{n\geqslant 1}$ is decreasing and that $\varepsilon_n < \frac{1}{2^{2n}}$ for all $n \in \mathbb{N}$. Since $(x_n)_{n\geqslant 1}$ is bounded away from every point in $[0,1]$ we may also assume that if $x_{k_n} \in [0,1]$ then $0 < x_{k_n} - \varepsilon_n < x_{k_n} + \varepsilon_n < 1$. Since locally we only sum over at most one term that is non-zero, the function $f : [0,1] \to [0,1]$ defined by

$$f(x) = \sum_{n:x_{k_n}\in[0,1]} \frac{1}{2^n} s_{x_n,\varepsilon_n}(x)$$

is well-defined, uniformly continuous and continuously differentiable on $[0,1]$. Furthermore

$$f' = \sum_{n:x_{k_n}\in[0,1]} \frac{1}{2^n} s'_{x_n,\varepsilon_n}$$

Now assume that f is uniformly continuously differentiable. Then its derivative f' would be bounded. So choose a natural number M such that $|f'| < M$. Assume there is $n \geqslant M$ such that $x_{k_n} \in [0,1]$. Equation (6) shows that there is a point $y \in [0,1]$ such that

$$s'_{x_{k_n},\varepsilon_n}(y) > \frac{1}{\varepsilon_n} > 2^{2n}.$$

Now

$$f'(y) = \frac{1}{2^n} s'_{x_{k_n},\varepsilon_n}(y) > \frac{2^{2n}}{2^n} > M;$$

a contradiction. Hence $x_{k_n} \notin [0,1]$ for every $n \geqslant M$. Since $(x_{k_n})_{n\geqslant 1}$ is also bounded away from 0 and 1 it is eventually bounded away from the entire interval $[0,1]$. By the properties of the chosen subsequence $(x_n)_{n\geqslant 1}$ is bounded away from the unit interval. □

The following implications hold:

$$\mathbf{UCT} \implies \mathbf{UDT} \implies \mathbf{AS}_{[0,1]}.$$

It remains an open question, whether any of the reverse implications hold. Notice that, to prove the reverse of the first implication, given a continuous function f one cannot simply apply **UDT** to a function F such that $F' = f$, since it is not clear, how to find such a function, without the knowledge that f is uniformly continuous.

Notice the following:

Proposition 19

(1) *If a function is uniformly differentiable, it is uniformly (actually even Lipschitz) continuous.*[4]

(2) *If a function is continuously differentiable, it is locally uniformly (actually even locally Lipschitz) continuous.*

Proof Simple consequence of the mean value theorem (Theorem 14). Consider $f : [a,b] \to \mathbb{R}$ continuously differentiable such that its derivative f is bounded. Hence we can find M such that $|f'| \leqslant M$. Now take any $x, y \in [a,b]$ with $x < y$. By Corollary 14 there exists $\xi \in [x,y]$ such that

$$\left| f'(\xi) - \frac{f(y) - f(x)}{y - x} \right| < 1.$$

Then

$$\left| \frac{f(y) - f(x)}{y - x} \right| \leqslant \left| f'(\xi) \right| + \left| f'(\xi) - \frac{f(y) - f(x)}{y - x} \right| < M + 1,$$

and therefore $|f(y) - f(x)| \leqslant (M+1)|x - y|$. By continuity, this holds for any x, y; and so f is Lipschitz continuous on $[a,b]$.

The same argument applies to a continuously differentiable function on a suitable sub-interval, since every continuous function is locally bounded. □

For integration we take the standard definition ([9]). That means that we have to be aware to integrate only uniformly continuous functions, because otherwise the integral is not well-defined.

Because we now have the Mean Value Theorem for continuously differentiable functions, we can expand the Fundamental Theorem of Calculus as found in [25] (Theorem 2.14) to get a result for continuously differentiable functions that is more comparable to Theorem 6.8 in [9].

Theorem 20 (Fundamental Theorem of Calculus) *Let $f : [0,1] \to \mathbb{R}$ be a uniformly continuous function, let $a \in [0,1]$, and write*

$$g(x) := \int_a^x f(t)dt.$$

Then g is uniformly differentiable and $g' = f$. Also, if g_0 is any differentiable function on $[0,1]$ with $g_0' = f$, then the difference $g - g_0$ is a constant function.

[4]This part of the proposition can already be found in [25, Proposition 6.2.2], where it is stated without proof.

The statement follows from Theorem 2.6.8 in [9] and Theorem 17. In [9] g_0 is taken to be *uniformly* differentiable, whereas in our version we assume continuous differentiability. Note that he proof of Theorem 20 makes an indirect use of the strong version of the mean value theorem (Corollary 14).

Theorem 2.14 of [25] does not contain any statement about differentiable functions of which the derivative equals f.

4 Picard's Theorem

Many variations of Picard's Theorem, which mainly differ in the level of abstractness, can be found in the literature. Because we work constructively, there is also an additional choice to make between classical equivalent formulations: Do we require the involved continuous functions to be uniformly continuous, or not?

In this section we will look at two choices. In the first, constructive version of Picard's Theorem we require the given function—the one that defines the differential equation—to be uniformly continuous.

Anticipating another version, in which we will not require the given function to be uniformly continuous, our formulation of the interval on which the solution can be found is vaguer than usual. Often that interval is characterised in terms of the supremum or an upper bound of the given function. Because it will not be clear later on that such a number exists, we are less distinctive about the size of the interval.

Theorem 21 (**Constructive Picard's Theorem**) *Let $a, b, c, d \in \mathbb{R}$; $(x_0, y_0) \in X = (a, b) \times (c, d)$, and $r > 0$ such that if $|x - x_0| \leqslant r$ and $|y - y_0| \leqslant r$, then $(x, y) \in [a, b] \times [c, d]$. Let $f : X \to \mathbb{R}$ be* uniformly *continuous, such that there exists $L > 0$ with*

$$|f(x, y_0) - f(x, y_1)| \leqslant L|y_0 - y_1|$$

for all applicable x, y_1, y_2. Then there exist a real number $h > 0$ and a unique uniformly *differentiable mapping ϕ on the interval $I = [x_0 - h, x_0 + h]$, such that*

$$\phi(x_0) = y_0$$

and

$$\phi'(x) = f(x, \phi(x)) \text{ for all } x \in I$$

Proof The standard proof applies (see e.g. [11]). We conclude that the solution is *uniformly* continuously differentiable by the Fundamental Theorem of Calculus (Theorem 20). □

Note that the version in [21], called the Cauchy/Lipschitz Theorem, is a weaker formulation than we have here. There the solution to the equation is not proven to have a *uniformly* continuous derivative.

The second version of Picard's Theorem requires only pointwise continuity for the defining function, and is hence stronger.

> **Strong Picard's Theorem** Let $a, b, c, d \in \mathbb{R}$; let $(x_0, y_0) \in X = (a, b) \times (c, d)$, and let $r > 0$ such that if $|x - x_0| \leqslant r$ and $|y - y_0| \leqslant r$, then $(x, y) \in [a, b] \times [c, d]$. Let $f : X \to \mathbb{R}$ be continuous, such that there exists $L > 0$ with
>
> $$|f(x, y_0) - f(x, y_1)| \leqslant L|y_0 - y_1|$$
>
> for all applicable x, y_1, y_2. Then there exist a real number $h > 0$ and a unique *uniformly* continuously differentiable mapping ϕ on the interval $I = [x_0 - h, x_0 + h]$, such that
>
> $$\phi(x_0) = y_0$$
>
> and
>
> $$\phi'(x) = f(x, \phi(x))\text{for all } x \in I.$$

This formulation bears some similarity to uniform continuity theorems: We start out with an pointwise continuous function and end up with a uniformly continuous one, although there it concerns exactly the same function. An additional similarity in the case of **LUCT** is that Picard's Theorem concludes uniform continuity only on subintervals.

Indeed it can be shown through **LUCT** that this, stronger, version of Picard's Theorem is equivalent to **UCT**.

Theorem 22 LUCT $\Leftrightarrow$ Strong Picard's Theorem

Proof To prove the direction $\Rightarrow$, assume **LUCT**. Determine $h > 0$ such that f is uniformly continuous on $[x_0 + h, x_0 - h]$. Now we can apply Constructive Picard's Theorem (Theorem 21). This gives us a uniformly continuously differential function ϕ on $[x_0 - h_1, x_0 + h_1]$ such that $\phi(x_0) = y_0$ and $\phi'(x) = f(x, \phi(x))$ for all $x \in [x_0 - h_1, x_0 + h_1]$.

We now prove the direction $\Leftarrow$. Let $f : [a, b] \to \mathbb{R}$ be a continuous function and let $x_0 \in [a, b]$. Define $g : [a - 1, b + 1] \times [0, 1]$ by

$$g(x, y) = \begin{cases} f(a) & \text{if } x \leqslant a \\ f(x) & \text{if } a \leqslant x \leqslant b \\ f(b) & \text{if } b \leqslant x \end{cases}$$

Then g is continuous by Lemma 2, and Lipschitz in the second variable. By **Strong Picard's Theorem**, the differential equation:

$$\phi(x_0) = 0$$

$$\phi'(x) = g(x, \phi(x))$$

has a uniformly continuously differential solution ϕ on an interval $[x_0 - h, x_0 + h]$. Because $\phi'(x) = f(x)$ on $[a, b]$, we now see that f is locally uniformly continuous. $\square$

Remark 23 *In the proof of* **LUCT** *out of* **Strong Picard's Theorem** *we have not used the fact that the solution is unique.*

5 Peano's Theorem

Although Picard's Theorem has thus a constructive core, the same cannot be said for Peano's Theorem.

> **Peano's Theorem** Let $a, b, c, d \in \mathbb{R}$; let $(x_0, y_0) \in X = (a, b) \times (c, d)$, and let $r > 0$ such that if $|x - x_0| \leqslant r$ and $|y - y_0| \leqslant r$, then $(x, y) \in [a, b] \times [c, d]$. Let $f : X \to \mathbb{R}$ be *uniformly* continuous; let
>
> $$M \geqslant \sup\{|f(x, y)| : |x - x_0| < r, \ |y - y_0| < r\},$$
>
> and let $h := \min\{r, rM^{-1}\}$. Then there exists a continuously differentiable mapping ϕ on the interval $I = [x_0 - h, x_0 + h]$, such that
>
> $$\phi(x_0) = y_0$$
>
> and
>
> $$\phi'(x) = f(x, \phi(x)) \text{ for all } x \in I.$$

This theorem is inherently nonconstructive: it is equivalent to the nonconstructive Lesser Limited Principle of Omniscience [4, 10]:

> **LLPO** For each binary sequence α with at most one term equal to 1, either $\alpha(2n) = 0$ for all n or $\alpha(2n + 1) = 0$ for all n.

It is instructive to look at the classical standard proof of Peano's Theorem and find out what goes "wrong" (see for example [11]). Given a (uniformly) continuous function f, a sequence of polynomial functions $(p_n)_{n \geqslant 1}$ is constructed that converges uniformly to it. Then, invoking Picard's Theorem and the Fundamental Theorem of Calculus, we find solutions $(\phi_n)_{n\geqslant 1}$ to the integral equation

$$y(x) = y_0 + \int_{x_0}^{x} p_n(t, y(t))dt.$$

Some calculations now show that the sequence $(\phi_n)_{n\geqslant 1}$ is bounded and equicontinuous. Applying Ascoli's Lemma, we now pass to a subsequence that converges uniformly to a limit ϕ. We conclude that ϕ is the solution to our original differential equation by some further calculations.

The main problem lies, of course, in the application of Ascoli's Lemma[5], which seems nonconstructive beyond repair, at least as far as finding a convergent subsequence is concerned ([23] and [14]).

To obtain a constructive version of Peano's Theorem, we therefore *assume* such a uniformly convergent subsequence. (In the classical proof the fact that this sequence originates from polynomial functions does not play a role after the application of Ascoli's Lemma. Note also that the equicontinuity of the sequence is only used to be able to apply Ascoli's Lemma and conclude uniform convergence, so we can dispense with that in the constructive version.)

Theorem 24 (Constructive Peano's Theorem) *Let* $a, b, c, d \in \mathbb{R}$*; let* $(x_0, y_0) \in (a, b) \times (c, d)$ *and define*

$$X = [a, b] \times [c, d].$$

Let $f : X \to \mathbb{R}$ *be* uniformly *continuous and let* $h > 0$*. There exists a* uniformly *convergent sequence of* uniformly *continuously differentiable functions* $(\phi_n)_{n\geqslant 1} : [x_0 - h, x_0 + h] \to \mathbb{R}$ *with* $\phi_n(x_0) = y_0$ *and such that for all* $\varepsilon > 0$ *there exists* $N \in \mathbb{N}$ *with*

$$\sup_{t\in[x_0-h, x_0+h]} |f(t, \phi_n(t)) - \phi_n'(t)| < \varepsilon$$

for all $n \geqslant N$

if and only if

there exists a uniformly *continuously differentiable function* ϕ *on the interval* $I = [x_0 - h, x_0 + h]$*, such that*

$$\phi(x_0) = y_0$$

[5] In classical Reverse mathematics, Peano's Theorem is equivalent to Weak König's Lemma over RCA_0. The proof in [21] avoids Ascoli's Lemma, which is classically equivalent to arithmetical comprehension, but is still non-constructive. We will not go into details.

and

$$\phi'(x) = f(x, \phi(x)) \text{ for all } x \in I.$$

Proof Assume that there exists $h > 0$ and a sequence $(\phi_n)_{\geqslant 1}$ with the required properties. Note that ϕ is a uniformly continuous function (by [14], Lemma 12). Let $n \in \mathbb{N}$. By uniform continuity of f, take $\delta > 0$ such that for each $(x_1, y_1), (x_2, y_2) \in X$, if $||(x_1, y_1), (x_2, y_2)|| < \delta$, then we can conclude that

$$|f(x_1, y_1) - f(x_2, y_2)| < 2^{-n}.$$

Choose N such that for all $m \geqslant N$

$$||\phi - \phi_m|| < \min\{\delta, 2^{-n}\},$$

where $||\ \ ||$ denotes the supremum norm, and

$$\sup_{t \in [x_0 - h, x_0 + h]} |f(t, \phi_m(t)) - \phi_m'(t)| < 2^{-n}.$$

We now have

$$\left| \int_{x_0}^{x} f(t, \phi(t))dt - \int_{x_0}^{x} f(t, \phi_N(t))dt \right| \leqslant 2^{-n}|x - x_0| < 2^{-n}|I|$$

and therefore

$$\begin{aligned}
\left| \phi(x) - y_0 - \int_{x_0}^{x} f(t, \phi(t))dt \right| \quad &\leqslant \quad |\phi(x) - \phi_N(x)| + \\
&\quad\ \ \left| \phi_N(x) - y_0 - \int_{x_0}^{x} \phi_N'(t)dt \right| + \\
&\quad\ \ \left| \int_{x_0}^{x} (\phi_N'(t) - f(t, \phi_N(t)))dt \right| + \\
&\quad\ \ \left| \int_{x_0}^{x} (f(t, \phi_N(t)) - f(t, \phi(t)))dt \right| \\
&< \quad 2^{-n} + 0 + |I|2^{-n} + |I|2^{-n} \\
&= \quad (1 + 2|I|)2^{-n}.
\end{aligned}$$

We conclude that

$$\phi(x) = y_0 + \int_{x_0}^{x} f(t, \phi(t))dt.$$

A final application of the Fundamental Theorem of Calculus (Theorem 20) shows that ϕ is uniformly continuously differentiable and satisfies the desired conditions.

To prove the other direction of the equivalence, suppose that we have a uniformly continuously differentiable solution $\phi : [x_0 - h, x_0 + h] \to \mathbb{R}$ to the differential equation.

Now take $(\phi_n)_{n\geqslant 1}$ to be the constant sequence defined by $\phi_n := \phi$ for every $n \geqslant 1$. It is easily seen that this sequence satisfies the requirements. □

Remark 25 *Note that the solution ϕ that we find in the proof of Theorem 24, is the limit of the sequence $(\phi_n)_{n\geqslant 1}$. So we have that*

$$(\lim_{n\to\infty} \phi_n)'(x) = f(x, (\lim_{n\to\infty} \phi_n)(x)) \text{ for all } x \in I.$$

It remains to be seen how useful this constructive version will be in practice. Given any function f, it is in general not possible to find such a sequence $(\phi_n)_{n\geqslant 1}$, as this would, again, imply **LLPO**. So the question is how we can restrict the (classical) theorem, such that such a sequence can be found. We will come back to this at the end of this section.

One case where it is possible, is when f is Lipschitz in the second variable, as in Constructive Picard's Theorem.

Lemma 26 *Let f be a function as in the assumptions of Constructive Picard's Theorem (Theorem 21). Then there exists a (non-trivial) uniformly convergent sequence of uniformly continuously differentiable functions $(\phi_n)_{n\geqslant 1} : [x_0 - h, x_0 + h] \to \mathbb{R}$ with $\phi_n(x_0) = y_0$ and such that for all $\varepsilon > 0$ there exists $N \in \mathbb{N}$ with*

$$\sup_{t\in[x_0-h,x_0+h]} |f(t, \phi_n(t)) - \phi_n'(t)| < \varepsilon$$

for all $n \geqslant N$.

Proof Let the sequence $(\phi_n)_{n\geqslant 1}$ on $[x_0 - h, x_0 + h]$ be defined by:

$$\phi_0(x) = y_0$$

and

$$\phi_{n+1}(x) = y_0 + \int_{x_0}^{x} f(t, \phi_n(t))dt$$

for all $n \geqslant 1$. Note that ϕ_n is uniformly continuously differentiable for each n by the Fundamental Theorem of Calculus (Theorem 20). See Exercise 4.7.5.4 of [11] for a sketch of a proof that this sequence converges uniformly and that $\phi := \lim_{n\to\infty} \phi_n$ is the solution to the differential equation. Let $\varepsilon > 0$. Determine $N \in \mathbb{N}$ such that for all $x \in [x_0 - h, x_0 + h]$ and for all $n \geqslant N$

$$|\phi(x) - \phi_n(x)| < L^{-1}\varepsilon.$$

Let $t \in [x_0 - h, x_0 + h]$ and $n \geqslant N$. Then

$$\begin{aligned} |f(t, \phi_{n+1}(t)) - \phi'_{n+1}(t)| &= |f(t, \phi_{n+1}(t) - f(t, \phi_n(t))| \\ &\leqslant L \cdot |\phi_{n+1}(t) - \phi_n(t)| \\ &< L \cdot L^{-1} \cdot \varepsilon = \varepsilon. \end{aligned}$$

Hence

$$\sup_{t \in [x_0 - h, x_0 + h]} |f(t, \phi_n(t)) - \phi'_n(t)| < \varepsilon$$

□

It now follows from Lemma 26 that, similar to the classical case, Constructive Picard's Theorem is a special case of Constructive Peano's Theorem, without the uniqueness of the result.

Next, we strengthen the constructive version of Peano's Theorem by neither requiring f to be uniformly continuous nor the ϕ_n's to be uniformly differentiable. We also replace 'uniformly convergent' by 'equicontinuous and convergent'.

> **Strong Peano's Theorem** Let $a, b, c, d \in \mathbb{R}$, $(x_0, y_0) \in (a, b) \times (c, d)$ and define
>
> $$X = [a, b] \times [c, d].$$
>
> Let $f : X \to \mathbb{R}$ be continuous and let $h > 0$. There exists an equicontinuous, convergent sequence of differentiable functions $(\phi_n)_{n \geqslant 1}$: $[x_0 - h, x_0 + h] \to \mathbb{R}$ with $\phi_n(x_0) = y_0$ and such that for all $\varepsilon > 0$ there exists $N \in \mathbb{N}$ with
>
> $$\sup_{t \in [x_0 - h, x_0 + h]} |f(t, \phi_n(t)) - \phi'_n(t)| < \varepsilon$$
>
> for all $n \geqslant N$
>
> if and only if
>
> there exists a *uniformly* continuously differentiable function ϕ on the interval $I = [x_0 - h, x_0 + h]$, such that
>
> $$\phi(x_0) = y_0$$
>
> and
>
> $$\phi'(x) = f(x, \phi(x)) \text{ for all } x \in I.$$

Theorem 27 **UCT** *and* **Strong Peano's Theorem** *are equivalent over* BISH.

Proof First assume **UCT**; we have to prove *Strong Peano's Theorem*. Let $f : X \to \mathbb{R}$ and let $h > 0$. Then f and each of the functions in the sequence $(\phi_n)_{n\geqslant 1}$ are uniformly continuous by **UCT**. It also follows from **UCT** that ϕ_n is uniformly continuously differentiable. Suppose that $(\phi_n)_{n\geqslant 1} : I \to \mathbb{R}$ is an equicontinuous, convergent sequence of differentiable functions with the properties as described in the theorem. Then $(\phi_n)_{n\geqslant 1}$ is uniformly convergent (by Theorem 18 of [14]). It now follows from Constructive Peano's Theorem (Theorem 24) that there exists a uniformly continuously differentiable solution ϕ to the differential equation.

To prove the other direction, we assume that there exists a uniformly continuously differentiable solution $\phi : I \to \mathbb{R}$ to the differential equation. Again we take

$$\phi_n := \phi$$

for all $n \geqslant 1$.

Now assume **Strong Peano's Theorem**; we have to prove **UCT**. Because **Strong Peano's Theorem** implies **Strong Picard's Theorem** without the uniqueness, **UCT** now follows by Theorem 22 and Remark 23. □

Let us now come back to the question how we can restrict Peano's Theorem in order for such a sequence $(\phi_n)_{n\geqslant 1}$ to be found. It is a general believe that classical existence theorems can be made constructive by requiering that any solution is (locally) unique. (See Bridges as quoted in [4] and [19].) It is therefore natural to consider a "uniqueness version" of Peano's Theorem, and to find out whether the condition that the differential equation has at most one solution enables us to find a sequence $(\phi_n)_{n\geqslant 1}$ constructively.

Aberth showed, however, in [1] that this is not possible by proving the existence of a differential equation as in Peano's Theorem without a computable solution.[6]

Some other, more recent papers [8, 7, 20] relate theorems with uniqueness conditions to versions of the Fan Theorem. One could say here that they show that uniqueness does not as much constructivise the theorems, but makes them intuitionistically valid. Bridges has shown in [10] that this also holds for Peano's Theorem. The Fan Theorem for decidable Bars ($\mathbf{FT_D}$) implies Peano's Theorem if we additionally assume that there exists at most one solution to the differential equation. It would be interesting to know whether "Unique Peano Theorem" is equivalent to $\mathbf{FT_D}$. Another open question is how $\mathbf{FT_D}$ enables us to construct a non-trivial sequence $(\phi_n)_{n\geqslant 1}$ given that there is at most one solution. See also the discussion in [10].

[6] Having no solution implies of course having at most one.

6 A recursive excursus

One of the stranger objects in the world of Russian recursive mathematics is "the" Specker sequence, that is a sequence in $[0,1]$ that is bounded away from every point in $[0,1]$.[7] Obviously, when such a sequence exists, $\mathbf{AS}_{[0,1]}$ fails to hold. Using the same construction as in Lemma 18, one can construct a bounded, continuous function, that fails to be uniformly continuous. Another such function, with a similar construction, can be found in [13]. Using this function and the same construction as in the proof of Proposition 7 we get the existence of a function $\tilde{s} : [0,1] \to \mathbb{R}$ that is not locally uniformly continuous. To be more precise $\tilde{s}$ fails to be uniformly continuous on any non-degenerate interval containing 1. This function already is a recursive counterexample to **Strong Picard's Theorem**.

Similarly, we can use the Specker sequence to turn the proof of Proposition 18 into a construction of a differentiable function that fails to be uniformly continuously differentiable.

There seems to be a general pattern here which one might like to call the constructive dialectic excursus: An equivalence to some version of the Fan theorem or **UCT** and a recursive counterexample all stemming from the same construction.

7 Conclusion and Discussion

Because the theorems in the field of differential equations that we have studied state the existence of a solution only on subintervals, we had to shift our attention from global to local properties. So instead of looking at uniformly continuous functions, we now used functions that are *locally* uniformly continuous. This has led to the identification of two new variants of the Uniform Continuity Theorem: The Uniform Continuity Theorem for Bounded Functions and the Locally Uniform Continuity Theorem.

Next we have reconsidered the definitions of differentiation that can be found in the literature. We have shown that pointwise differentiability is a useful notion by proving Rolle's theorem, the mean value theorem and a version of the fundamental theorem of calculus for pointwise (or: continuously) differentiable functions.

After that we have considered two ways to bring a notion of uniformity into the definition of differentiation. The first one, uniform differentiability, is well-known and seemed

[7] The original Specker sequence constructed in [22], and other such sequences, employs even stronger properties—they are increasing sequences of rational numbers.

in first instance stronger than the new notion of uniformly continuous differentiability. By applying the new theorems on pointwise differentiability we were, however, able to demonstrate that these two uniformity notions are equivalent.

Then we have defined the Uniform Differentiation Theorem and placed it into the hierarchy of fan theorems and associated notions. It turned out to be in between **UCT** and $\mathbf{AS}_{[a,b]}$, the latter of which is equivalent to the fan theorem for c-bars. The uniform continuity theorem and the fan theorem for c-bars seem already very close, so it might be slightly surprising that anything can fit between them. It is therefore hoped that **UDT** will turn out to be equivalent to one of them.

Finally Picard's Theorem and Peano's Theorem, two existence theorems in the field of differential equations, were studied in the light of constructive reverse mathematics. Picard's Theorem has a constructive core and we have seen both a constructive version of it and a version that we proved equivalent to the Locally Uniform Continuity Theorem. Peano's theorem is essentially non-constructive. By a careful examination of the standard proof we were able to formulate a much weaker constructive version, and one that we have also shown to be equivalent to the Uniform Continuity Theorem.

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