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Projective maximal families of orthogonal measures with large continuum

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Abstract: We study maximal orthogonal families of Borel probability measures on 2^ω (abbreviated m.o. families) and show that there are generic extensions of the constructible universe L in which each of the following holds:

(1) There is a Δ^1_3-definable well-ordering of the reals, there is a Π^1_2-definable m.o. family, there are no $\mathbf{\Sigma^1_2}$-definable m.o. families and $\mathfrak{b} = \mathfrak{c} = \omega_3$ (in fact any reasonable value of $\mathfrak{c}$ will do).

(2) There is a Δ^1_3-definable well-ordering of the reals, there is a Π^1_2-definable m.o. family, there are no $\mathbf{\Sigma^1_2}$-definable m.o. families, $\mathfrak{d} = \omega_1$ and $\mathfrak{c} = \omega_2$.

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1 Introduction

Let X be a Polish space, and let $P(X)$ denote the Polish space of Borel probability measures on X, in the sense of [9, 17.E]. Recall that if $\mu, \nu \in P(X)$ then μ and ν are said to be *orthogonal*, written $\mu \perp \nu$, if there is a Borel set $B \subseteq X$ such that $\mu(B) = 0$ and $\nu(X \setminus B) = 0$. A set of measures $\mathcal{A} \subseteq P(X)$ is said to be *orthogonal* if whenever $\mu, \nu \in \mathcal{A}$ and $\mu \neq \nu$ then $\mu \perp \nu$. A *maximal orthogonal family*, or *m.o. family*, is an orthogonal family $\mathcal{A} \subseteq P(X)$ which is maximal under inclusion.

The present paper is concerned with the study of *definable* m.o. families. A well-known result due to Preiss and Rataj [13] states that there are no analytic m.o. families. In a recent paper [3] it was shown by Fischer and Törnquist that if all reals are constructible then there is a Π^1_1 m.o. family. The latter paper also raised the question how restrictive the existence of a definable m.o. family is on the structure of the real line, since it was shown that Π^1_1 m.o. families cannot coexist with Cohen reals.

In the present paper we study Π^1_2 m.o. families in the context of $\mathfrak{c} \geq \omega_2$, with the additional requirement that there is a Δ^1_3-definable wellorder of $\mathbb{R}$. Our main results are:

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Theorem 1 It is consistent with $\mathfrak{c} = \mathfrak{b} = \omega_3$ that there is a Δ^1_3-definable wellorder of the reals, a Π^1_2 definable maximal orthogonal family of measures and there are no $\mathbf{\Sigma^1_2}$-definable maximal sets of orthogonal measures.

There is nothing special about $\mathfrak{c} = \omega_3$: The same result can be obtained for any reasonable value of $\mathfrak{c}$.

Theorem 2 It is consistent with $\mathfrak{d} = \omega_1$, $\mathfrak{c} = \omega_2$ that there is a Δ^1_3-definable wellorder of the reals, a Π^1_2 definable maximal orthogonal family of measures and there are no $\mathbf{\Sigma^1_2}$-definable maximal sets of orthogonal measures.

Taken together these theorems indicate that the existence of a Π^1_2 m.o. family does not seem to impose any severe restrictions on the structure of the real line. On the other hand, we show (Proposition 1) that $\mathbf{\Sigma^1_2}$ m.o. families cannot coexist with either Cohen or random reals, extending the previous result of Fischer and Törnquist that Π^1_1 m.o. families cannot coexist with Cohen reals. This is the explanation why in the models produced to prove Theorems 1 and 2 there are no $\mathbf{\Sigma^1_2}$ m.o. families.

The theorems of this paper belong to a line of results concerning the definability of certain combinatorial objects on the real line and in particular the question of how low in the projective hierarchy such objects exist. In [12] Mathias showed that there is no $\mathbf{\Sigma}^1_1$-definable maximal almost disjoint (mad) family in $[\omega]^\omega$. Assuming $V = L$, Miller obtained a Π^1_1 mad family in $[\omega]^\omega$, see [11].

The study of the existence of definable combinatorial objects on $\mathbb{R}$ in the presence of a projective wellorder of the reals and $\mathfrak{c} \geq \omega_2$ was initiated in [1], [4] and [2]. The wellorder of $\mathbb{R}$ in all those models has a Δ^1_3-definition, which is indeed optimal for models of $\mathfrak{c} \geq \omega_2$, since by Mansfield's theorem (see [7, Theorem 25.39]) the existence of a Σ^1_2-definable wellorder of the reals implies that all reals are constructible. The existence of a Π^1_2-definable ω-mad family in $[\omega]^\omega$ in the presence of $\mathfrak{c} = \mathfrak{b} = \omega_2$ was established by Friedman and Zdomskyy in [4]. In the same paper, referring to earlier results (see [14] and [8]) they outlined the construction of a model in which $\mathfrak{c} = \omega_2$ and there is a Π^1_1-definable ω-mad family: Start with the constructible universe L, obtain a Π^1_1-definable ω mad family and proceed with a countable support iteration of length ω_2 of Miller forcing. The techniques were further developed in [2] to establish a model in which there is a Π^1_2-definable ω-mad family and $\mathfrak{c} = \mathfrak{b} = \omega_3$. In particular, in the models from [4] and [2], there are no maximal almost disjoint families of size $< \mathfrak{c}$ and so the almost disjointness number has a Π^1_2-witness.

The present paper combines the encoding techniques of [3] with the techniques of [1, 4, 2] to obtain Theorems 1 and 2. We note that one significant difference from the situation for mad families is that m.o. families always have size $\mathfrak{c}$ (see [3, Proposition 4.1]). Moreover, owing to the fact that our coding technique for measures (Lemma 1) preserves the measure class, the forcing constructions in §3 and §4 is somewhat simplified compared to [1, 4, 2].

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2 Preliminaries

In this section, we briefly recall the coding of probability measures on 2^ω and the encoding technique for measures introduced in [3].

Let X be a Polish space. Recall that if $\mu, \nu \in P(X)$ then μ is said to be *absolutely continuous* with respect to ν, written $\mu \ll \nu$, if for all Borel subsets of X we have that $\nu(B) = 0$ implies that $\mu(B) = 0$. Two measures $\mu, \nu \in P(2^\omega)$ are called *absolutely equivalent*, written $\mu \approx \nu$, if $\mu \ll \nu$ and $\nu \ll \mu$.

If $s \in 2^{<\omega}$ we let $N_s = \{x \in 2^\omega : s \subseteq x\}$ be the basic neighbourhood determined by s. Following [3], we let

$$p(2^\omega) = \{f : 2^{<\omega} \to [0,1] : f(\emptyset) = 1 \wedge (\forall s \in 2^{<\omega}) f(s) = f(s^\frown 0) + f(s^\frown 1)\}.$$

The spaces $p(2^\omega)$ and $P(2^\omega)$ are homeomorphic via the recursive isomorphism $f \mapsto \mu_f$ where $\mu_f \in P(2^\omega)$ is the measure uniquely determined by requiring that $\mu_f(N_s) = f(s)$ for all $s \in 2^{<\omega}$. We call the unique real $f \in p(2^\omega)$ such that $\mu = \mu_f$ the *code* for μ. The identification of $P(2^\omega)$ and $p(2^\omega)$ allows us to use the notions of effective descriptive set theory in the space $P(2^\omega)$. For instance, the set $P_c(2^\omega)$ of all non-atomic probability measures on 2^ω is arithmetical because the set $p_c(2^\omega) = \{f \in p(2^\omega) : \mu_f \text{ is non-atomic}\}$ is easily seen to be arithmetical, as shown in [3].

We will use the method of coding a real $z \in 2^\omega$ into a measure $\mu \in P_c(2^\omega)$ introduced in [3]. For convenience we recall the construction in minimal detail. Given $\mu \in P_c(2^\omega)$ and $s \in 2^{<\omega}$ we let $t(s,\mu)$ be the lexicographically least $t \in 2^{<\omega}$ such that $s \subseteq t$, $\mu(N_{t^\frown 0}) > 0$ and $\mu(N_{t^\frown 1}) > 0$, if it exists and otherwise we let $t(s,\mu) = \emptyset$. Define recursively $t^\mu_n \in 2^{<\omega}$ by letting $t^\mu_0 = \emptyset$ and $t^\mu_{n+1} = t(t^{\mu\frown}_n 0, \mu)$. Since μ is non-atomic, we have $\mathrm{lh}(t^\mu_{n+1}) > \mathrm{lh}(t^\mu_n)$. Let $t^\mu_\infty = \bigcup_{n=0}^\infty t^\mu_n$. For $f \in p_c(2^\omega)$ and $n \in \omega \cup \{\infty\}$ we will write t^f_n for $t^{\mu_f}_n$. Clearly the sequence $(t^f_n : n \in \omega)$ is recursive in f.

Define the relation $R \subseteq p_c(2^\omega) \times 2^\omega$ as follows:

$$R(f,z) \iff (\forall n \in \omega)\big(z(n) = 1 \longleftrightarrow (f(t^{f\frown}_n 0) = \frac{2}{3} f(t^f_n) \wedge f(t^{f\frown}_n 1) = \frac{1}{3} f(t_n))\big)$$
$$\wedge \big(z(n) = 0 \leftrightarrow f(t^{f\frown}_n 0) = \frac{1}{3} f(t^f_n) \wedge f(t^{f\frown}_n 1) = \frac{2}{3} f(t^f_n)\big).$$

Whenever $(f,z) \in R$ we say that f *codes* z. Note that $\mathrm{dom}(R) = \{f \in p_c(2^\omega) : (\exists z) R(f,z)\}$ is Π^0_1 and so the function $r : \mathrm{dom}(R) \to 2^\omega$, where $r(f) = z$ if and only if $(f,z) \in R$, is also Π^0_1. The key properties of this construction is contained in the following Lemma (see [3, Coding Lemma]):

Lemma 1 There is a recursive function $\bar r : p_c(2^\omega) \times 2^\omega \to p_c(2^\omega)$ such that $\mu_{\bar r(f,z)} \approx \mu_f$ and $R(\bar r(f,z), z)$ for all $f \in p_c(2^\omega)$ and $z \in 2^\omega$.

The proofs of Theorems 1 and 2 use the following result, which we now prove.

Proposition 1 Let $a \in \mathbb{R}$ and suppose that there either is a Cohen real over $L[a]$ or there is a random real over $L[a]$. Then there is no $\Sigma^1_2(a)$ m.o. family.

We first need a preparatory Lemma. In 2^ω, consider the equivalence E_I defined by

$$xE_Iy \iff \sum_{n=0}^{\infty} \frac{|x(n)-y(n)|}{n+1} < \infty.$$

We identify 2^ω with $\mathbb{Z}_2^\omega$ and equip it with the Haar measure μ.

Lemma 2 Let $A \subseteq 2^\omega$ be a Borel set such that $\mu(A) > 0$. Then $E_I \leq_B E_I \upharpoonright A$, where $E_I \upharpoonright A$ is the restriction of E_I to A.

Notation: The constant 0 sequence of length $n \in \omega \cup \{\infty\}$ is denoted 0^n. If $A \subseteq 2^\omega$ and $s \in 2^{<\omega}$ let

$$A_{(s)} = \{x \in 2^\omega : s^\frown x \in A\},$$

the *localization* of A at s.

Proof of Lemma 2 We may assume that $A \subseteq 2^\omega$ is closed. We will define $q_n \in \omega$, $s_{n,i}, s_t \in 2^{<\omega}$ recursively for all $n \in \omega$, $i \in \{0,1\}$ and $t \in 2^{<\omega}$ satisfying

(1) $q_0 = 0$ and $q_{n+1} = q_n + \mathrm{lh}(s_{n,0})$.

(2) $s_{0,i} = \emptyset$ and $\mathrm{lh}(s_{n,i}) = \mathrm{lh}(s_{n,1-i}) > 0$ when $n > 0$.

(3) $s_\emptyset = \emptyset$ and $s_{t^\frown i} = s_t{}^\frown s_{\mathrm{lh}(t)+1,i}$ for all $t \in 2^{<\omega}$, $i \in \{0,1\}$.

(4) $\frac{1}{n+1} \leq \sum_{k=0}^{\mathrm{lh}(s_{n+1,0})} \frac{|s_{n+1,0}(k)-s_{n+1,1}(k)|}{q_n+k+1} \leq \frac{2}{n+1}$.

(5) $N_{s_t} \subseteq A$.

(6) If $t \in 2^n$ then $\mu(A_{(s_t)}) > 1 - 2^{-n}$.

Suppose this can be done. We claim that the map $2^\omega \to A : x \mapsto a_x$ defined by

$$a_x = \bigcup_{n\in\omega} s_{x\upharpoonright n}$$

is a Borel (in fact, continuous) reduction of E_I to $E_I \upharpoonright A$. To see this, fix $x, y \in 2^\omega$ and note that by (4) we have that

$$\sum_{n=0}^{\infty} \frac{|x(n)-y(n)|}{n+1} \leq \sum_{n=0}^{\infty} \sum_{k=0}^{\mathrm{lh}(s_{n+1,0})} \frac{|s_{n+1,x(i)}(k) - s_{n+1,y(i)}(k)|}{q_n+k+1} = \sum_{n=0}^{\infty} \frac{|a_x(n)-a_y(n)|}{n+1} \leq 2\sum_{n=0}^{\infty} \frac{|x(n)-y(n)|}{n+1}$$

so that xE_Iy if and only if $a_xE_Ia_y$.

We now show that we can construct a scheme satisfying (1)–(6) above. Suppose q_k, $s_{k,i}$ and s_t have been defined for all $k \leq n$ and $t \in 2^{\leq n}$. It is enough to define $s_{n+1,i}$ satisfying (4)–(6). Define

$$f_{q_n} : 2^\omega \to [0,\infty] : f_{q_n}(x) = \sum_{k=0}^{\infty} \frac{x(k)}{q_n+k+1}.$$

It is clear that $f_{q_n}(N_{0^k})$ is dense in $[0,\infty]$ for all $k \in \omega$. Let

$$A' = \{x \in A : \lim_{k\to\infty} \mu(A_{(x\restriction k)}) \to 1\},$$

i.e, the set of points in A of density 1. By the Lebesgue density theorem [9, 17.9] we have $\mu(A \setminus A') = 0$. Let $A'' = \bigcap_{t\in 2^n} A'_{(s_t)}$ and note that by (6) we have $\mu(A'') > 0$. Thus the set of differences $A'' - A''$ contains a neighborhood of 0^∞ by [9, 17.13]. It follows that there are $x_0, x_1 \in A''$ such that

$$\frac{1}{n+2} \leq \sum_{k=0}^{\infty} \frac{|x_0(k) - x_1(k)|}{q_n + k + 1} \leq \frac{2}{n+2}.$$

Since all points in $A'_{(s_t)}$ have density 1 in $A'_{(s_t)}$ there is some $k_0 \in \omega$ such that

$$\mu(A'_{(s_t{}^\frown x_i\restriction k_0)}) > 1 - 2^{-n-1}$$

for all $t \in 2^n$. Defining $s_{n+1,i} = x_i \restriction k_0$, it is then clear that (4)–(6) holds. □

Proof of Proposition 1 As the proof easily relativizes, assume that $a = 0$. We proceed exactly as in [3, Proposition 4.2]. Suppose $A \subseteq P(2^\omega)$ is a Σ^1_2 m.o. family. Recall from [10] and [3, p. 1406] that there is a Borel function $2^\omega \to P(2^\omega) : x \mapsto \mu^x$ such that

$$xE_I y \implies \mu^x \approx \mu^y$$

and

$$x \not{E}_I y \implies \mu^x \perp \mu^y.$$

Define as in [3, Proposition 4.2] a relation $Q \subseteq 2^\omega \times P(2^\omega)^\omega$ by

$$Q(x,(\nu_n)) \iff (\forall n)(\nu_n \in A \wedge \nu_n \not\perp \mu^x) \wedge (\forall \mu)(\mu \not\perp \mu^x \longrightarrow (\exists n)\nu_n \not\perp \mu)$$

and note that this is Σ^1_2 when A is. Note that $Q(x,(\nu_n))$ precisely when (ν_n) enumerates the measures in A not orthogonal to μ^x (this set is always countable, see [10, Theorem 3.1].) Since A is maximal, each section Q_x is non-empty, and so we can uniformize Q with a (total) function $f : 2^\omega \to p(2^\omega)^\omega$ having a Δ^1_2 graph. Note that assignment

$$x \mapsto A(x) = \{f(x)_n : n \in \mathcal{N}\}$$

is invariant on the E_I classes.

If there is a Cohen real over L it follows from [6] that f is Baire measurable. Since E_I is a turbulent equivalence relation (in the sense of Hjorth, see e.g. [10]) the map $x \mapsto A(x)$ must be constant on a comeagre set. But this contradicts that all E_I classes are meagre.

If on the other hand there is a random real over L, then f is Lebesgue measurable by [6]. Let $F \subseteq 2^\omega$ be a closed set with positive measure on which f is continuous, and let $g : 2^\omega \to F$ be a Borel reduction of E_I to $E_I \restriction F$. Note that $x \mapsto A(g(x))$ is then an E_I-invariant Borel assignment of countable subsets of $p(2^\omega)$, and so since E_I is turbulent the function $f \circ g$ must be constant on a comeagre set. This again contradicts that all E_I classes are meagre. □

3 Δ^1_3 w.o. of the reals, Π^1_2 m.o. family, no $\mathbf{\Sigma}^1_2$ m.o. families with $\mathfrak{b} = \mathfrak{c} = \omega_3$

We proceed with the proof of Theorem 1. We will use a modification of the model constructed in [2]. We work over the constructible universe L. Recall that a transitive ZF^- model is *suitable* if $\omega_3^{\mathcal{M}}$ exists and $\omega_3^{\mathcal{M}} = \omega_3^{L^{\mathcal{M}}}$. If $\mathcal{M}$ is suitable then also $\omega_1^{\mathcal{M}} = \omega_1^{L^{\mathcal{M}}}$ and $\omega_2^{\mathcal{M}} = \omega_2^{L^{\mathcal{M}}}$. Our construction can be considered a two stage process - a preliminary stage and a coding stage. In the preliminary stage (Steps 0 through 3 below), we obtain a generic extension of L over which we can perform a finite support iteration of length ω_3 (coding stage), leading to a model satisfying Theorem 1.

Fix a $\diamondsuit_{\omega_2}(cof(\omega_1))$ sequence $\langle G_\xi : \xi \in \omega_2 \cap cof(\omega_1)\rangle$ which is Σ_1-definable over L_{ω_2}. For $\alpha < \omega_3$, let W_α be the L-least subset of ω_2 coding α and for $1 < \alpha < \omega_3$ let $S_\alpha = \{\xi \in \omega_2 \cap cof(\omega_1) : G_\xi = W_\alpha \cap \xi \neq \emptyset\}$. Then $\vec{S} = \langle S_\alpha : 1 < \alpha < \omega_3\rangle$ is a sequence of stationary subsets of $\omega_2 \cap cof(\omega_1)$, which are mutually almost disjoint. Let $S_{-1} = \{\xi \in \omega_2 \cap \mathrm{cof}(\omega_1) : G_\xi = \emptyset\}$. Note that S_{-1} is a stationary subset of $\omega_2 \cap \mathrm{cof}(\omega_1)$ which is disjoint from all S_α's.

Step 0. For every α such that $\omega_2 \leq \alpha < \omega_3$ shoot a club C_α disjoint from S_α via the poset $\mathbb{P}^0_\alpha$, consisting of all closed subsets of ω_2 which are disjoint from S_α with the extension relation being end-extension, and let $\mathbb{P}^0 = \prod_{\alpha<\omega_3} \mathbb{P}^0_\alpha$ be the direct product of the $\mathbb{P}^0_\alpha$'s with supports of size ω_1, where for $\alpha \in \omega_2$, $\mathbb{P}^0_\alpha$ is the trivial poset. Then $\mathbb{P}^0$ is countably closed, ω_2-distributive (the proof of which uses the stationarity of S_{-1}) and ω_3-c.c.

Step 1. We begin by fixing some notation. Let $Lim'(\omega_2)$ be the set of all limit ordinals ξ in ω_2 which can be presented in the form $\xi = \omega \cdot \omega \cdot \alpha''$ for some $\alpha'' \geq 0$. Let $Lim'(\omega_3)$ be the set of all limit ordinals α in ω_3 which can be presented in the form $\alpha = \omega^2 \cdot \alpha' + \xi$, where $\alpha' > 0$ and $\xi \in Lim'(\omega_2)$. Also, whenever $k \in \omega$, X is a set of ordinals and $j \in k$, let $I^k_j(X) = \{\gamma : k \cdot \gamma + j \in X\}$. In particular, let $Even(X) = I^2_0(X) = \{\gamma : 2 \cdot \gamma \in X\}$.

Let $\alpha \in [\omega_2, \omega_3)$. Then $\alpha = \alpha_0 + \omega \cdot k + m$ for some $\alpha_0 \in \mathrm{Lim}'(\omega_3)$, $k, m \in \omega$. Then, let $D_\alpha = D^k_\alpha$ be a subset of ω_2 coding the tuple $(C_\alpha, W_\alpha, \langle W_{\alpha_0+\omega\cdot j}\rangle_{j\in k+1})$. More precisely, let $D_\alpha = D^k_\alpha$ be a subset of ω_2 such that $I^{k+3}_j(D_\alpha) = W_{\alpha_0+\omega\cdot j}$ for $j \in k+1$, $I^{k+3}_{k+1}(D_\alpha) = D_\alpha$ and $I^{k+3}_{k+2}(D_\alpha) = C_\alpha$. Now let

$$E_\alpha = E^k_\alpha = \{\mathcal{M} \cap \omega_2 : \mathcal{M} \prec L_{\alpha+\omega_2+1}[D_\alpha], \omega_1 \cup \{D_\alpha\} \subseteq \mathcal{M}\}.$$

Then E_α is a club on ω_2. Choose $Z_\alpha = Z^k_\alpha \subseteq \omega_2$ such that $Even(Z_\alpha) = D_\alpha$ and if $\beta < \omega_2$ is the $\omega_2^{\mathcal{M}}$ for some suitable model $\mathcal{M}$ such that $Z_\alpha \cap \beta \in \mathcal{M}$, then $\beta \in E_\alpha$. Then we have:

$(*)_{\alpha,k}$: If $\beta < \omega_2$, $\mathcal{M}$ is a suitable model such that $\omega_1 \subset \mathcal{M}$, $\omega_2^{\mathcal{M}} = \beta$, and $Z_\alpha \cap \beta \in \mathcal{M}$, then $\mathcal{M} \vDash \psi_k(\omega_2, Z_\alpha \cap \beta)$, where $\psi_k(\omega_2, X)$ is the formula "$Even(X)$ codes a triple $(\bar{C}, \bar{W}, \langle \bar{W}_j\rangle_{j\in k+1})$, where $\bar{W}$ and $\bar{W}_k$ are the L-least codes of ordinals $\bar{\alpha}, \bar{\alpha}_k < \omega_3$ such that $\bar{\alpha}_k$ is the largest limit ordinal not exceeding $\bar{\alpha}$, for $j \in k$ $\bar{W}_j$ is the L-least code for the largest limit ordinal $\bar{\alpha}_j$ strictly smaller than $\bar{\alpha}_{j+1}$, and $\bar{C}$ is a club in ω_2 disjoint from $S_{\bar{\alpha}}$".

Similarly to $\vec{S}$, define a sequence $\vec{A} = \langle A_\xi : \xi < \omega_2 \rangle$ of stationary subsets of ω_1 using the "standard" $\diamondsuit$-sequence. Code Z_α by a subset $X_\alpha = X_\alpha^k$ of ω_1 with the poset $\mathbb{P}^1_\alpha$ consisting of all pairs $\langle s_0, s_1 \rangle \in [\omega_1]^{<\omega_1} \times [Z_\alpha]^{<\omega_1}$ where $\langle t_0, t_1 \rangle \leq \langle s_0, s_1 \rangle$ iff s_0 is an initial segment of t_0, $s_1 \subseteq t_1$ and $t_0 \backslash s_0 \cap A_\xi = \emptyset$ for all $\xi \in s_1$. Then X_α satisfies the following condition:

$(**)_{\alpha,k}$: If $\omega_1 < \beta \leq \omega_2$ and $\mathcal{M}$ is a suitable model such that $\omega_2^{\mathcal{M}} = \beta$ and $\{X_\alpha\} \cup \omega_1 \subset \mathcal{M}$, then $\mathcal{M} \vDash \phi_k(\omega_1, \omega_2, X_\alpha)$, where $\phi_k(\omega_1, \omega_2, X)$ is the formula: " Using the sequence $\vec{A}$, X almost disjointly codes a subset $\bar{Z}$ of ω_2, such that $Even(\bar{Z})$ codes a triple $(\bar{C}, \bar{W}, \langle \bar{W}_j \rangle_{j \in k+1})$, where $\bar{W}$ and $\bar{W}_k$ are the L-least codes of ordinals $\bar{\alpha}, \bar{\alpha}_k < \omega_3$ such that $\bar{\alpha}_k$ is the largest limit ordinal not exceeding $\bar{\alpha}$, for $j \in k$ $\bar{W}_j$ is the L-least code for the largest limit ordinal $\bar{\alpha}_j$ strictly smaller than $\bar{\alpha}_{j+1}$, and $\bar{C}$ is a club in ω_2 disjoint from $S_{\bar{\alpha}}$".

Let $\mathbb{P}^1 = \prod_{\alpha < \omega_3} \mathbb{P}^1_\alpha$, where $\mathbb{P}^1_\alpha$ is the trivial poset for all $\alpha \in \omega_2$, with countable support. Then $\mathbb{P}^1$ is countably closed and has the ω_2-c.c.

Finally we force a localization of the X_α's. Fix ϕ_k as in $(**)_{\alpha,k}$ and define the poset $\mathcal{L}_k(X, X')$ similarly to the poset defined in [2, Definition 1] as follows.

Definition 3.1 Let $X, X' \subset \omega_1$ be such that $\phi_k(\omega_1, \omega_2, X)$ and $\phi_k(\omega_1, \omega_2, X')$ hold in any suitable model $\mathcal{M}$ with $\omega_1^{\mathcal{M}} = \omega_1^L$ containing X and X', respectively. Then let $\mathcal{L}_k(X, X')$ be the poset of all functions $r : |r| \to 2$, where the domain $|r|$ of r is a countable limit ordinal such that:

(1) if $\gamma < |r|$ then $\gamma \in X$ iff $r(3\gamma) = 1$

(2) if $\gamma < |r|$ then $\gamma \in X'$ iff $r(3\gamma + 1) = 1$

(3) if $\gamma \leq |r|$, $\mathcal{M}$ is a countable suitable model containing $r \upharpoonright \gamma$ as an element and $\gamma = \omega_1^{\mathcal{M}}$, then $\mathcal{M} \vDash \phi_k(\omega_1, \omega_2, X \cap \gamma) \wedge \phi_k(\omega_1, \omega_2, X' \cap \gamma)$.

The extension relation is end-extension.

For every $\alpha \in Lim'(\omega_3)$, $k, m \in \omega$, let $\mathbb{P}^2_{\alpha,k,m} = \mathcal{L}_k(X_{\alpha+\omega\cdot k+m}, X_{\alpha+\omega\cdot k})$ and for $\alpha \in \omega_2$, let $\mathbb{P}^2_\alpha$ be the trivial poset. Let

$$\mathbb{P}^2 = (\prod_{\alpha \in Lim'(\omega_3)} \prod_{k,m \in \omega} \mathbb{P}^2_{\alpha,k,m}) \times (\prod_{\alpha \in \omega_2} \mathbb{P}^2_\alpha)$$

with countable supports. Note that the poset $\mathbb{P}^2_{\alpha,k,m}$, where $\alpha \in Lim'(\omega_3)$, $k, m \in \omega$, produces a generic function in ${}^{\omega_1}2$ (of $L^{\mathbb{P}^0 * \mathbb{P}^1}$), which is the characteristic function of a subset $Y_{\alpha,k,m}$ of ω_1 with the following property:

$(***)_{\alpha,k}$: For every $\beta < \omega_1$ and any suitable $\mathcal{M}$ such that $\omega_1^{\mathcal{M}} = \beta$ and $Y_{\alpha,k,m} \cap \beta$ belongs to $\mathcal{M}$, we have $\mathcal{M} \vDash \phi_k(\omega_1, \omega_2, X_{\alpha+\omega\cdot k+m} \cap \beta) \wedge \phi_k(\omega_1, \omega_2, X_{\alpha+\omega\cdot k} \cap \beta)$.

Similarly to the proof of [2, Lemma 1] one can show that $\mathbb{P}_0 := \mathbb{P}^0 * \mathbb{P}^1 * \mathbb{P}^2$ is ω-distributive.

Step 3. We proceed with the coding stage of our construction. We will define a finite support iteration $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta : \alpha \leq \omega_3, \beta < \omega_3 \rangle$ such that $\mathbb{P}_0 = \mathbb{P}^0 * \mathbb{P}^1 * \mathbb{P}^2$, for every $\alpha < \omega_3$, $\dot{\mathbb{Q}}_\alpha$ is a $\mathbb{P}_\alpha$-name for a σ-centered poset, in $L^{\mathbb{P}_{\omega_3}}$ there is a Δ^1_3-definable wellorder of the reals, a Π^1_2-definable maximal family of orthogonal measures, there are no $\boldsymbol{\Sigma^1_2}$-definable maximal families of orthogonal measures and $\mathfrak{b} = \mathfrak{c} = \omega_3$. Along the iteration for every $\alpha < \omega_3$, we will define in $V^{\mathbb{P}_\alpha}$ a set O_α of codes for measures and a set A_α of ordinals. Every $\mathbb{Q}_\alpha$ will add a generic real, whose $\mathbb{P}_\alpha$-name will be denoted $\dot{u}_\alpha$ and similarly to the proof of [2, Lemma 2] one can prove that $L[G_\alpha] \cap {}^\omega\omega = L[\langle \dot{u}_\xi^{G_\alpha} : \xi < \alpha \rangle] \cap {}^\omega\omega$ for every $\mathbb{P}_\alpha$-generic filter G_α. This gives a canonical wellorder of the reals in $L[G_\alpha]$ which depends only on the sequence $\langle \dot{u}_\xi : \xi < \alpha \rangle$, whose $\mathbb{P}_\alpha$-name will be denoted by $\dot{<}_\alpha$. We can additionally arrange that for $\alpha < \beta$, $<_\alpha$ is an initial segment of $<_\beta$, where $<_\alpha = \dot{<}_\alpha^{G_\alpha}$ and $<_\beta = \dot{<}_\beta^{G_\beta}$. Then if G is a $\mathbb{P}_{\omega_3}$-generic filter over L, then $<^G = \bigcup \{ \dot{<}_\alpha^G : \alpha < \omega_3 \}$ will be the desired wellorder of the reals.

We will need some more notation. If x, y are reals in $L[G_\alpha]$ such that $x <_\alpha y$, let $x * y = \{2n : n \in x\} \cup \{2n+1 : n \in y\}$ and $\Delta(x * y) = \{2n+2 : n \in x * y\} \cup \{2n+1 : n \notin x * y\}$. For every $\alpha \in [\omega_2, \omega_3)$, let $\dot{F}^0_\alpha$, $\dot{F}^1_\alpha$ be $\mathbb{P}_\alpha$-names for nicely definable bijections $F^0_\alpha : 2^\omega \to p_c(2^\omega)$ and $F^1_\alpha : (2^\omega)^\omega \to 2^\omega$ in $L[G_\alpha]$, respectively, such that whenever $\alpha < \beta$, $i \in \{0, 1\}$ we have $F^i_\alpha \subseteq F^i_\beta$. For example, identifying $p_c(2^\omega)$ with countable sequences of reals, let $(F^0_\alpha)^{-1}$, F^1_α be simply Cantor diagonalization. For every $\nu \in [\omega_2, \omega_3)$ let $i_\nu : \nu \cup \{\langle \xi, \eta \rangle : \xi < \eta < \nu\} \to Lim'(\omega_2)$ be a fixed bijection and let $\vec{B} = \langle B_{\zeta,m} : \zeta < \omega_1, m \in \omega \rangle$ be a nicely definable sequence of almost disjoint subsets of ω.

Suppose $\mathbb{P}_\alpha$ has been defined and fix a $\mathbb{P}_\alpha$-generic filter G_α.

Case A. Suppose $\alpha \in Lim'(\omega_3)$, i.e. $\alpha = \omega^2 \cdot \alpha' + \xi$ for some $\alpha' > 0$, $\xi \in Lim'(\omega_2)$. Let $\nu = o.t.(\dot{<}^{G_\alpha}_{\omega_2 \cdot \alpha'})$, $i = i_\nu$.

Case A.1. If $i^{-1}(\xi) = \langle \xi_0, \xi_1 \rangle$ for some $\xi_0 < \xi_1 < \nu$, let x_{ξ_0} and x_{ξ_1} be the ξ_0-th and ξ_1-th reals in $L[G_{\omega_2 \cdot \alpha'}]$ according to the wellorder $\dot{<}^{G_\alpha}_{\omega_2 \cdot \alpha'}$. In $L^{\mathbb{P}_\alpha}$ let

$$\mathbb{Q}_\alpha = \{ \langle s_0, s_1 \rangle : s_0 \in [\omega]^{<\omega}, s_1 \in [\bigcup_{m \in \Delta(x_{\xi_0} * x_{\xi_1})} Y_{\alpha+m} \times \{m\}]^{<\omega} \},$$

where $\langle t_0, t_1 \rangle \leq \langle s_0, s_1 \rangle$ if and only if $s_1 \subseteq t_1$, s_0 is an initial segment of t_0 and $(t_0 \backslash s_0) \cap B_{\zeta,m} = \emptyset$ for all $\langle \zeta, m \rangle \in s_1$. Let u_α be the generic real added by $\mathbb{Q}_\alpha$, $A_\alpha = \alpha + \omega \backslash \Delta(x_{\xi_0} * x_{\xi_1})$ and $O_\alpha = \emptyset$. For every $n \geq 1$, let $\mathbb{Q}_{\alpha+n}$ be the poset (in $L^{\mathbb{P}_{\alpha+n}}$) adding a dominating real $u_{\alpha+n}$, which is defined in *Case B* below and let $A_{\alpha+n} = O_{\alpha+n} = \emptyset$.

Case A.2. Suppose $i^{-1}(\xi) = \zeta \in \nu$. Consider the ζ-th real x_ζ according to the wellorder $\dot{<}^{G_\alpha}_{\omega_2 \cdot \alpha'}$. Let $F^0_{\omega_2 \cdot \alpha'} = \dot{F}^0_{\omega_2 \cdot \alpha'}[G_\alpha]$ and let $f_\alpha = (F^0_{\omega_2 \cdot \alpha'})(x_\zeta)$.

Case A.2.1. If f_α is not a code for a measure orthogonal to $O'_\alpha = \bigcup_{\gamma<\alpha} O_\gamma$, for every $n \in \omega$ recursively define in $L^{\mathbb{P}_{\alpha+n}}$, $\mathbb{Q}_{\alpha+n}$ to be the poset for adding a dominating real defined in *Case B* below and let $A_{\alpha+n} = O_{\alpha+n} = \emptyset$.

Case A.2.2. Otherwise, i.e. in case f_α is a code for a measure orthogonal to $O'_\alpha = \bigcup_{\gamma<\alpha} O_\gamma$, define the poset $\mathbb{Q}_{\alpha+n}$, the set of codes for measures $O_{\alpha+n}$ and the set of ordinals $A_{\alpha+n}$ in $L^{\mathbb{P}_{\alpha+n}}$ recursively as follows.

- $\mathbb{Q}_\alpha$ almost disjointly, via the sequence $\vec{B}$, codes the sequence $\langle Y_{\alpha+m} : m \in \Delta(x_\zeta)\rangle$. More precisely let
$$\mathbb{Q}_\alpha = \{\langle s_0, s_1\rangle : s_0 \in [\omega]^{<\omega}, s_1 \in [\bigcup_{m\in\Delta(x_\zeta)} Y_{\alpha+m} \times \{m\}]^{<\omega}\},$$
where $\langle t_0, t_1\rangle \leq \langle s_0, s_1\rangle$ if and only if $s_1 \subseteq t_1$, s_0 is an initial segment of t_0 and $(t_0\backslash s_0) \cap B_{\zeta,m} = \emptyset$ for all $\langle \zeta, m\rangle \in s_1$. Let u_α be the generic real added by $\mathbb{Q}_\alpha$, $A_\alpha = \alpha + \omega\backslash\Delta(u_\alpha)$.
- Let $n \geq 1$. Suppose $\mathbb{Q}_{\alpha+(n-1)}$ has been defined and adds a real $u_{\alpha+(n-1)}$. Then $\mathbb{Q}_{\alpha+n}$ almost disjointly, via the sequence $\vec{B}$, codes the sequence $\langle Y_{\alpha+\omega\cdot n+m} : m \in \Delta(u_{\alpha+(n-1)})\rangle$. More precisely let
$$\mathbb{Q}_{\alpha+n} = \{\langle s_0, s_1\rangle : s_0 \in [\omega]^{<\omega}, s_1 \in [\bigcup_{m\in\Delta(u_{\alpha+(n-1)})} Y_{\alpha+\omega\cdot n+m} \times \{m\}]^{<\omega}\},$$
where $\langle t_0, t_1\rangle \leq \langle s_0, s_1\rangle$ if and only if $s_1 \subseteq t_1$, s_0 is an initial segment of t_0 and $(t_0\backslash s_0)\cap B_{\zeta,m} = \emptyset$ for all $\langle \zeta, m\rangle \in s_1$. Let $u_{\alpha+n}$ be the generic real added by $\mathbb{Q}_{\alpha+n}$, $A_{\alpha+n} = \alpha + \omega\cdot n + \omega\backslash\Delta(u_{\alpha+(n-1)})$.

In $L^{\mathbb{P}_{\alpha+\omega}}$ let $\vec{u}_\alpha = (u^\alpha_n)_{n\in\omega}$, where $u^\alpha_0 = x_\zeta$ and $u^\alpha_n = u_{\alpha+n-1}$ whenever $n \geq 1$. Let
$$g_\alpha = \bar{r}(F^0_{\alpha+\omega}(u^\alpha_0), F^1_{\alpha+\omega}((u^\alpha_n)_{n\geq 1}))$$
(see Lemma 1) and for every $n \in \omega$ let $O_{\alpha+n} = \{g_\alpha\}$.

Case B. Suppose either $\alpha \in \omega_2$, or $\alpha \in Lim(\omega_3)\backslash Lim'(\omega_3)$, or α is a successor ordinal in (ω_2, ω_3) which is not of the form $\alpha' + n$ for some $\alpha' \in Lim'(\omega_3)$, $n \in \omega$. Then let $\mathbb{Q}_\alpha$ be the following poset for adding a dominating real:
$$\mathbb{Q}_\alpha = \{\langle s_0, s_1\rangle : s_0 \in \omega^{<\omega}, s_1 \in [\text{o.t.}(\dot{<}^{G_\alpha}_\alpha)]^{<\omega}\},$$
where $\langle t_0, t_1\rangle \leq \langle s_0, s_1\rangle$ if and only if s_0 is an initial segment of t_0, $s_1 \subseteq t_1$, and $t_0(n) > x_\xi(n)$ for all $n \in \text{dom}(t_0)\backslash\text{dom}(s_0)$ and $\xi \in s_1$, where x_ξ is the ξ-th real in $L[G_\alpha] \cap \omega^\omega$ according to the wellorder $\dot{<}^{G_\alpha}_\alpha$. Let $A_\alpha = \emptyset$, $O_\alpha = \emptyset$.

With this the definition of $\mathbb{P}_{\omega_3}$ is complete. Similarly to [2, Lemma 3] one can show that if G is $\mathbb{P}_{\omega_3}$-generic and $\xi \in \bigcup_{\alpha\in\omega_3} \dot{A}^G_\alpha$, then in $L[G]$ there is no real coding a stationary kill of S_ξ. We will refer to this fact, as *no accidental coding of stationary kill*. Also, in $L^{\mathbb{P}_{\omega_3}}$, let $O = \bigcup_{\alpha\in\omega_3} O_\alpha$, $F^0 = \bigcup_{\alpha\in\omega_3} F^0_\alpha$, $F^1 = \bigcup_{\alpha\in\omega_3} F^1_\alpha$ and for $\vec{z} = (z_n)_{n\in\omega}$, let $\mathcal{R}(\vec{z}) = \bar{r}(F^0(z_0), F^1((z_n)_{n\geq 1}))$ (see Lemma 1).

Lemma 3.2 *Let G be $\mathbb{P}_{\omega_3}$ generic, $g = \mathcal{R}(\vec{z})$ for some $\vec{z} = (z_n)_{n\in\omega}$. Then $g \in O$ if and only if for every countable suitable model $\mathcal{M}$ such that $g \in \mathcal{M}$ there is $\bar{\alpha} < \omega_3^{\mathcal{M}}$ such that for all $n \in \omega$, $S_{\bar{\alpha}+\omega\cdot n+m}$ is non-stationary in $(L[z_{n+1}])^{\mathcal{M}}$ for all $m \in \Delta(z_n)$.*

Proof Suppose $g = \mathcal{R}(\vec{z})$ and for every countable suitable model $\mathcal{M}$ such that $g \in \mathcal{M}$, there is $\bar{\alpha} < \omega_3^{\mathcal{M}}$ with the above property. By the Löweinheim-Skolem theorem, the same holds in $\mathbb{H}_\Theta^{\mathbb{P}}$, where Θ is sufficiently large and $\mathbb{H}_\Theta^{\mathbb{P}}$ denotes the set of all sets hereditary of cardinality $< \Theta$. Thus there is $\alpha < \omega_3$ such that for all $n \in \omega$, $m \in \Delta(z_n)$, $L_\Theta[z_{n+1}] \vDash (S_{\alpha+\omega\cdot n+m}$ is not stationary$)$. Then in particular for some $n > 0$, $m \in \omega$ the stationary kill of $S_{\alpha+\omega\cdot n+m}$ is coded by a real. Since there is no accidental coding of stationary kill, $\mathbb{Q}_{\alpha+n}$ adds a real $u_{\alpha+n}(= u_{n+1}^\alpha)$ coding a stationary kill of $S_{\alpha+\omega\cdot n+m}$ for all $m \in \Delta(u_n^\alpha)$, while there are no reals coding the stationary kill of $S_{\alpha+\omega\cdot n+m}$ for $m \notin \Delta(u_n^\alpha)$. Therefore $\Delta(u_n^\alpha) = \Delta(z_n)$ for all n, and so $\vec{u}_\alpha = \vec{z}$, which implies $g = g_\alpha \in O$.

On the other hand, suppose $g \in \mathcal{R}(\vec{z}) \in O$. Thus $g = g_\alpha = \mathcal{R}(\vec{u}_\alpha)$ and since $\mathcal{R}$ is injective $\vec{u}_\alpha = \vec{z}$. Suppose $\mathcal{M}$ is a suitable model which contains g. Then by definition of $\bar{r}$ we have that $F^0(u_0^\alpha)$ and $F^1((u_n^\alpha)_{n\geq 1})$ are also in $\mathcal{M}$. Since F^0, F^1 are nicely definable, $\mathcal{M}$ contains also $\vec{u}_\alpha$. Therefore for all $n \in \omega$, $m \in \Delta(u_n^\alpha)$ also the sets $Y_{\alpha+\omega\cdot n+m} \cap \omega_1^{\mathcal{M}}$ are in $\mathcal{M}$. Thus in particular, $\mathcal{M}$ contains the sets $X_{\alpha+\omega\cdot n+m} \cap \omega_1^{\mathcal{M}}$ for all $n \in \omega$, $m \in \Delta(u_n^\alpha)$. Fix $n, m \in \Delta(u_n^\alpha)$. By definition of $\mathcal{L}_n(X_{\alpha+\omega\cdot n+m}, X_{\alpha+\omega\cdot n})$ we have that for every $m \in \Delta(u_n^\alpha)$, in $\mathcal{M}$, using the sequence $\vec{A}$, the set $X_{\alpha+\omega\cdot n+m} \cap \omega_1$ almost disjointly codes a subset $\bar{Z}^{n,m}$ of ω_2, whose even part codes a triple $(C^{n,m}, W^{n,m}, \langle W_j^{n,m}\rangle_{j\in n+1}\rangle)$, where $W^{n,m}$, $W_n^{n,m}$ are the L-least codes of ordinals $\alpha^{n,m}$, $\alpha_n^{n,m}$ in ω_3 such that $\alpha_n^{n,m}$ is the largest limit ordinal not exceeding $\alpha^{n,m}$ and for every $j \in n$, $\alpha_j^{n,m}$ is the largest limit ordinal strictly smaller than $\alpha_{j+1}^{m,n}$. It remains to observe that for every $n_1 < n_2$, m_1, m_2 in ω, we have $W_j^{n_1,m_1} = W_j^{n_2,m_2}$ whenever $j \leq n_1$. Therefore $\alpha_0^{n,m}$ does not depend on n, m and so $\bar{\alpha} = \alpha_0^{n,m}$ is the desired ordinal. □

Therefore O has indeed a Π_2^1 definition. We will show that O is maximal in $p_c(2^\omega)$. Indeed, suppose in $L^{\mathbb{P}_{\omega_3}}$ there is a code f for a measure orthogonal to every measure in the family $\bar{O} = \{\mu_g : g \in O\}$. Choose α minimal such that $\alpha = \omega_2 \cdot \alpha' + \xi$ for some $\alpha' > 0$, $\xi \in Lim'(\omega_2)$ and such that $f \in L[G_{\omega_2\cdot\alpha'}]$. Let $\nu = o.t.(\dot{<}_{\omega_2\cdot\alpha'}^{G_\alpha})$ and let $i = i_\nu$. Then $x = (F_{\omega_2\cdot\alpha'}^0)^{-1}(f)$ is the ζ-th real according to the wellorder $\dot{<}_{\omega_2\cdot\alpha'}^{G_\alpha}$ for some $\zeta \in \nu$, which implies that for some $\xi \in Lim'(\omega_2)$, $i^{-1}(\xi) = \zeta$. But then $x_\zeta = x$ is the code of a measure orthogonal to O'_α and so by construction O_α contains the code of a measure equivalent to μ_f, which is a contradiction. To obtain a Π_2^1-definable m.o. family in $L^{\mathbb{P}_{\omega_3}}$ consider the union of $\bar{O} = \{\mu_g : g \in O\}$ with the set of all point measures. Just as in [2] one can show that $<$ is indeed a Δ_3^1-definable wellorder of the reals.

Since $\mathbb{P}_{\omega_3}$ is a finite support iteration, we have added Cohen reals along the iteration cofinally often. Thus for every real a in $L^{\mathbb{P}_{\omega_3}}$ there is a Cohen real over $L[a]$ and so by Proposition 1 in $L^{\mathbb{P}_{\omega_3}}$ there are no $\boldsymbol{\Sigma_2^1}$ m.o. families. Also note that since cofinally often we have added dominating reals, $L^{\mathbb{P}_{\omega_3}} \vDash \mathfrak{b} = \omega_3$.

4 Δ^1_3 w.o. of the reals, a Π^1_2 m.o. family, no $\mathbf{\Sigma^1_2}$ m.o. families with $\mathfrak{d} = \omega_1$ and $\mathfrak{c} = \omega_2$

In this section we establish the proof of Theorem 2. The model is obtained as a modification of the iteration construction developed in [1]. We restate the definitions of the posets used in this construction. For a more detailed account of their properties see [1]. We work over the constructible universe L. For the remainder of this section, we will say that a transitive ZF^- model is *suitable*, if $\omega_2^{\mathcal{M}}$ exists and $\omega_2^{\mathcal{M}} = \omega_2^{\mathcal{M}^L}$.

If $S \subseteq \omega_1$ is a stationary, co-stationary set, then by $Q(S)$ denote the poset of all countable closed subsets of $\omega_1 \backslash S$ with the extension relation being end-extension. Recall that $Q(S)$ is $\omega_1 \backslash S$-proper, ω-distributive and adds a club disjoint from S (see [1], [5]). For the proof of Theorem 2 we use the form of localization defined in [1, Definition 1]. That is, if $X \subseteq \omega_1$ and $\phi(\omega_1, X)$ is a Σ_1-sentence with parameters ω_1, X which is true in all suitable models containing ω_1 and X as elements, then let $\mathcal{L}(\phi)$ be the poset of all functions $r : |r| \to 2$, where the domain $|r|$ of r is a countable limit ordinal, such that

(1) if $\gamma < |r|$ then $\gamma \in X$ iff $r(2\gamma) = 1$

(2) if $\gamma \leq |r|$, $\mathcal{M}$ is a countable, suitable model containing $r \restriction \gamma$ as an element and $\gamma = \omega_1^{\mathcal{M}}$, then $\phi(\gamma, X \cap \gamma)$ holds in $\mathcal{M}$.

The extension relation is end-extension. Recall that $\mathcal{L}(\phi)$ has a countably closed dense subset (see [1, Remark 2]) and that if G is $\mathcal{L}(\phi)$-generic and $\mathcal{M}$ is a countable suitable model containing $(\bigcup G) \restriction \gamma$ as an element, where $\gamma = \omega_1^{\mathcal{M}}$, then $\mathcal{M} \vDash \phi(\gamma, X \cap \gamma)$ (see [1, Lemma 2]).

We will use also the coding with perfect trees defined in [1, Definition 2]. Let $Y \subseteq \omega_1$ be generic over L such that in $L[Y]$ cofinalities have not been changed and let $\bar{\mu} = \{\mu_i\}_{i \in \omega_1}$ be a sequence of L-countable ordinals such that μ_i is the least $\mu > \sup_{j<i} \mu_j$, $L_\mu[Y \cap i] \vDash ZF^-$ and $L_\mu \vDash \omega$ is the largest cardinal. Say that a real R *codes* Y below i if for all $j < i$, $j \in Y$ if and only if $L_{\mu_j}[Y \cap j, R] \vDash ZF^-$. For $T \subseteq 2^{<\omega}$ a perfect tree, let $|T|$ be the least i such that $T \in L_{\mu_i}[Y \cap i]$. Then $\mathcal{C}(Y)$ is the poset of all perfect trees T such that R codes Y below $|T|$, whenever R is a branch through T, where for T_0, T_1 conditions in $\mathcal{C}(Y)$, $T_0 \leq T_1$ if and only if T_0 is a subtree of T_1. Recall also that $\mathcal{C}(Y)$ is proper and ${}^\omega\omega$-bounding (see [1, Lemmas 7,8]).

Fix a bookkeeping function $F : Lim'(\omega_2) \to L_{\omega_2}$ and a sequence $\vec{S} = (S_\beta : \beta < \omega_2)$ of almost disjoint stationary subsets of ω_1, defined as in [1, Lemma 14]. Thus F and $\vec{S}$ are Σ_1-definable over L_{ω_2} with parameter ω_1, $F^{-1}(a)$ is unbounded in $Lim'(\omega_2)$ for every $a \in L_{\omega_2}$ and whenever $\mathcal{M}, \mathcal{N}$ are suitable models such that $\omega_1^{\mathcal{M}} = \omega_1^{\mathcal{N}}$ then $F^{\mathcal{M}}, \vec{S}^{\mathcal{M}}$ agree with $F^{\mathcal{N}}, \vec{S}^{\mathcal{N}}$ on $\omega_2^{\mathcal{M}} \cap \omega_2^{\mathcal{N}}$. Also if $\mathcal{M}$ is suitable and $\omega_1^{\mathcal{M}} = \omega_1$ then $F^{\mathcal{M}}, \bar{S}^{\mathcal{M}}$ equal the restrictions of F, $\vec{S}$ to the ω_2 of $\mathcal{M}$. Fix also a stationary subset S of ω_1 which is almost disjoint from every element of $\vec{S}$.

Recursively we will define a countable support iteration $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta : \alpha \leq \omega_2, \beta < \omega_2 \rangle$ such that in $L^{\mathbb{P}_{\omega_2}}$ there is a Δ^1_3-definable wellorder of the reals, there is a Π^1_2 definable m.o. family, there are no $\mathbf{\Sigma^1_2}$ definable m.o. families and $\mathfrak{d} = \aleph_1$, $\mathfrak{c} = \aleph_2$. Along the iteration for every $\alpha < \omega_2$, we will define in $L^{\mathbb{P}_\alpha}$ a set O_α of codes for measures and a set A_α of ordinals. Define the wellorder $<_\alpha$ in $L[G_\alpha]$ where G_α is $\mathbb{P}_\alpha$-generic just as in [1]. We can assume that all names for reals are nice and that for $\alpha < \beta < \omega_2$, all $\mathbb{P}_\alpha$-names for reals precede in the canonical wellorder $<_L$ of L all $\mathbb{P}_\beta$-names for reals, which are not $\mathbb{P}_\alpha$-names. For each $\alpha < \omega_2$, define a wellorder $<_\alpha$ on the reals of $L[G_\alpha]$, where G_α is a $\mathbb{P}_\alpha$-generic as follows. If x is a real in $L[G_\alpha]$ let σ^α_x be the $<_L$-least $\mathbb{P}_\gamma$-name for x, where $\gamma \leq \alpha$ is least so that x has a $\mathbb{P}_\gamma$-name. For x, y reals in $L[G_\alpha]$ define $x <_\alpha y$ if and only if $\sigma^\alpha_x <_L \sigma^\alpha_y$. Note that whenever $\alpha < \beta$, then $<_\alpha$ is an initial segment of $<_\beta$. Then $<^G = \bigcup_{\alpha<\omega_2} \dot{<}^G_\alpha$ will be the desired wellorder of the reals in $L[G]$, where G is a $\mathbb{P}_{\omega_2}$-generic filter. For every $\alpha \in \omega_2$, let W_α be the L-least subset of ω_1 coding α. Also, for each $\alpha \in \omega_2$, fix $\mathbb{P}_\alpha$-names $\dot{F}^0_\alpha$, $\dot{F}^1_\alpha$ for nicely definable bijections $F^0_\alpha : 2^\omega \to p_c(2^\omega)$, $F^1_\alpha : (2^\omega)^\omega \to 2^\omega$ in $L^{\mathbb{P}_\alpha}$ such that for all $i \in \{0, 1\}$ and $\alpha < \beta < \omega_2$ in $L^{\mathbb{P}_\beta}$ we have $F^i_\alpha \subseteq F^i_\beta$ (e.g. take $(F^0_\alpha)^{-1}$, F^1_α to be Cantor diagonalization).

We proceed with the definition of the poset. Let $\mathbb{P}_0$ be the trivial poset. Suppose $\mathbb{P}_\alpha$, $\langle O_\gamma : \gamma < \alpha\rangle$ and $\langle A_\gamma : \gamma < \alpha\rangle$ have been defined. Let G_α be a $\mathbb{P}_\alpha$-generic filter.

Case A. Suppose $\alpha \in Lim'(\omega_2) = \{\alpha \in Lim(\omega_2) : \alpha = \omega\cdot\omega\cdot\alpha''$ for some $\alpha'' \geq 0\}$. We will define $\mathbb{P}_{\alpha+\gamma}$ for $\gamma \in \omega\cdot\omega$ recursively as follows.

Case A.1. Suppose $F(\alpha) = \{\sigma^\alpha_x, \sigma^\alpha_y\}$ is a pair of nice names for reals in $L[G_\alpha]$. Let $x = \sigma^\alpha_x[G_\alpha]$, $y = \sigma^\alpha_y[G_\alpha]$.

- For every $m \in \omega$, define $\mathbb{Q}_{\alpha+m}$ in $L^{\mathbb{P}_{\alpha+m}}$ as follows. If $m \in \Delta(x * y)$ let $\mathbb{Q}_{\alpha+m} = Q(S_{\alpha+m})$ and if $m \notin \Delta(x * y)$ let $\mathbb{Q}_{\alpha+m}$ be the random real forcing.
- In $L^{\mathbb{P}_{\alpha+\omega}}$ let $X_{\alpha+\omega}$ be a subset of ω_1, coding W_α, coding the pair (x, y), coding a level of L in which α has size at most ω_1 and coding the generic $G_{\alpha+\omega}$, which can be regarded as a subset of an element of L_{ω_2}. Let $\mathbb{K}^1_{\alpha+\omega} = \mathcal{L}(\phi_{\alpha+\omega})$, where $\phi_{\alpha+\omega} = \phi_{\alpha+\omega}(\omega_1, X_{\alpha+\omega})$ is the Σ_1-sentence which holds if and only if $X_{\alpha+\omega}$ codes a subset W of ω_1 and a pair (x, y) of reals, such that W is the L-least code for an ordinal $\bar{\alpha} < \omega_2$ and $S_{\bar{\alpha}+m}$ is non-stationary for $m \in \Delta(x * y)$. Let $\dot{X}_{\alpha+\omega}$ be a $\mathbb{P}_{\alpha+\omega}$-name for $X_{\alpha+\omega}$ and let $\dot{\mathbb{K}}^1_{\alpha+\omega}$ be a $\mathbb{P}_{\alpha+\omega}$-name for $\mathbb{K}^1_{\alpha+\omega}$.
- Let $Y_{\alpha+\omega}$ be $\mathbb{K}^1_{\alpha+\omega}$-generic over $L[G_{\alpha+\omega}]$. The even part of $Y_{\alpha+\omega}$ codes $X_{\alpha+\omega}$ and so codes the generic $G_{\alpha+\omega}$. Then in $L[Y_{\alpha+\omega}] = L[G_{\alpha+\omega} * Y_{\alpha+\omega}]$, let $\mathbb{K}^2_{\alpha+\omega} = \mathcal{C}(Y_{\alpha+\omega})$. Let $R_{\alpha+\omega}$ be the real added by $\mathbb{K}^2_{\alpha+\omega}$, let $\dot{\mathbb{K}}^2_{\alpha+\omega}$ be a $\mathbb{P}_{\alpha+\omega} * \dot{\mathbb{K}}^1_{\alpha+\omega}$-name for $\mathbb{K}^2_{\alpha+\omega}$ and let $\mathbb{Q}_{\alpha+\omega} = \mathbb{K}^1_{\alpha+\omega} * \dot{\mathbb{K}}^2_{\alpha+\omega}$.
- For every $\gamma \in [\alpha + \omega + 1, \alpha + \omega\cdot\omega)$ let $\mathbb{Q}_{\alpha+\gamma}$ be a $\mathbb{P}_{\alpha+\gamma}$-name for the random real forcing.

Case A.2. Suppose $F(\alpha) = \{\sigma^\alpha_x\}$ for some nice name for a real σ^α_x. Let $x = \sigma^\alpha_x[G_\alpha]$, $f = F^0_\alpha(x)$.

Case A.2.1. If f is not a code of a measure orthogonal to $O'_\alpha = \bigcup_{\gamma<\alpha} O_\gamma$, let $\mathbb{Q}_{\alpha+\gamma}$ be a $\mathbb{P}_{\alpha+\gamma}$-name for the random real forcing, for all $\gamma \in \omega\cdot\omega$.

Case A.2.2. If f is a code of a measure orthogonal to $O'_\alpha = \bigcup_{\gamma<\alpha} O_\gamma$, define $\mathbb{Q}_{\alpha+\gamma}$ for $\gamma \in \omega\cdot\omega$ recursively as follows. Let $\mathbb{Q}_\alpha$ be the trivial poset (in $L^{\mathbb{P}_\alpha}$), and let $R_\alpha = x$. Suppose the poset $\mathbb{P}_{\alpha+\omega\cdot n+1}$ and the real $R_{\alpha+\omega\cdot n}$ have been defined.

- For $m \geq 1$ define $\mathbb{Q}_{\alpha+\omega\cdot n+m}$ in $L^{\mathbb{P}_{\alpha+\omega\cdot n+m}}$ recursively as follows. If $m-1 \in \Delta(R_{\alpha+\omega\cdot n})$ let $\mathbb{Q}_{\alpha+\omega\cdot n+m} = Q(S_{\alpha+\omega\cdot n+(m-1)})$ and if $m-1 \in \Delta(R_{\alpha+\omega\cdot n})$ let $\mathbb{Q}_{\alpha+\omega\cdot n+m}$ be the random real forcing.
- Let $G_{\alpha+\omega\cdot n+\omega}$ be a $\mathbb{P}_{\alpha+\omega\cdot n+\omega}$-generic filter. In $L[G_{\alpha+\omega\cdot n+\omega}]$ let $X_{\alpha+\omega\cdot n+\omega}$ be a subset of ω_1 coding $W_{\alpha+\omega\cdot j}$ for $j \leq n+1$, coding the real $R_{\alpha+\omega\cdot n}$, coding a level of L in which $\alpha+\omega\cdot n+\omega$ has size at most ω_1 and coding the generic $G_{\alpha+\omega\cdot n+\omega}$. Let $\mathbb{K}^1_{\alpha+\omega\cdot(n+1)}$ be the poset $\mathcal{L}(\phi^{n+1}_\alpha)$, where $\phi^{n+1}_\alpha = \phi^{n+1}_\alpha(\omega_1, X_{\alpha+\omega\cdot(n+1)})$ is the Σ_1-sentence which holds if and only if $X_{\alpha+\omega\cdot(n+1)}$ codes the tuple $\langle \bar{W}_j\rangle_{j\leq n+1}$ of subsets of ω_1 and a real x, such that $\bar{W}_{n+1}$ is the L-least code for an ordinal $\bar{\alpha} = \bar{\alpha}_{n+1}$ and for every $j \leq n$, $\bar{W}_j$ is the L-least code for the largest limit $\bar{\alpha}_j$ strictly smaller than $\bar{\alpha}_{j+1}$, and for every $m \in \Delta(x)$, the set $S_{\bar{\alpha}+m}$ is non-stationary. Let $\dot{X}_{\alpha+\omega\cdot(n+1)}$ be a $\mathbb{P}_{\alpha+\omega\cdot(n+1)}$-name for $X_{\alpha+\omega\cdot(n+1)}$, $\dot{\mathbb{K}}^1_{\alpha+\omega\cdot(n+1)}$ is a $\mathbb{P}_{\alpha+\omega\cdot(n+1)}$-name for $\mathbb{K}^1_{\alpha+\omega\cdot(n+1)}$.
- Let $Y_{\alpha+\omega\cdot(n+1)}$ be $\mathbb{K}^1_{\alpha+\omega\cdot(n+1)}$-generic filter over $L[G_{\alpha+\omega\cdot(n+1)}]$. In $L[Y_{\alpha+\omega\cdot(n+1)}]$ (note that the even part of $Y_{\alpha+\omega\cdot(n+1)}$ codes $X_{\alpha+\omega\cdot(n+1)}$ and so the generic $G_{\alpha+\omega\cdot(n+1)}$) let $\mathbb{K}^2_{\alpha+\omega\cdot(n+1)} = \mathcal{C}(Y_{\alpha+\omega\cdot(n+1)})$ and let $R_{\alpha+\omega\cdot(n+1)}$ be the generic real added by $\mathbb{K}^2_{\alpha+\omega\cdot(n+1)}$. Let $\mathbb{Q}_{\alpha+\omega\cdot(n+1)} = \mathbb{K}^1_{\alpha+\omega\cdot(n+1)} * \dot{\mathbb{K}}^2_{\alpha+\omega\cdot(n+1)}$.

In $L^{\mathbb{P}_{\alpha+\omega\cdot\omega}}$, let $u^\alpha_n = R_{\alpha+\omega\cdot n}$ for $n \in \omega$ (in particular $u^\alpha_0 = x$.) Let $\vec{u}_\alpha = (u^\alpha_n)_{n\in\omega}$ and let

$$g_\alpha = \bar{r}(F^0_{\alpha+\omega\cdot\omega}(u^\alpha_0), F^1_{\alpha+\omega\cdot\omega}((u^\alpha_n)_{n\geq 1}))$$

(see Lemma 1). For every $\gamma \in [\alpha, \alpha+\omega\cdot\omega)$ let $O_\gamma = \{g_\alpha\}$. For $n \in \omega$, let $A_{\alpha+\omega\cdot n} = \alpha+\omega\cdot n+\omega \backslash \Delta(u^\alpha_n)$ and for γ successor in $[\alpha, \alpha+\omega\cdot\omega)$, let $A_\gamma = \emptyset$.

Case B. Suppose $\alpha \in Lim(\omega_2)\backslash Lim'(\omega_2)$, or α is a successor ordinal in ω_2 which can not be presented in the form $\alpha'+\omega\cdot n+m$ for some $\alpha' \in Lim'(\omega_2)$, $n, m \in \omega$. Then let $\dot{\mathbb{Q}}_\alpha$ be a $\mathbb{P}_\alpha$-name for the random real forcing. Let $O_\alpha = A_\alpha = \emptyset$.

With this the recursive construction of $\mathbb{P}_{\omega_2}$ is complete. Similarly to [1, Lemma 17], one can show that if G is $\mathbb{P}_{\omega_2}$-generic and $\xi \in \bigcup_{\xi\in\omega_2} \dot{A}^G_\xi$, then S_ξ is stationary in $L[G]$. We will refer to this fact as *no accidental stationary kill*. In $L^{\mathbb{P}_{\omega_2}}$, let $O = \bigcup_{\alpha<\omega_2} O_\alpha$, $F^0 = \bigcup_{\alpha\in\omega_2} F^0_\alpha$, $F^1 = \bigcup_{\alpha\in\omega_2} F^1_\alpha$ and for $\vec{z} = (z_n)_{n\in\omega} \in (2^\omega)^\omega$ let $\mathcal{R}(\vec{z}) = \bar{r}(F^0(z_0), F^1((z_n)_{n\geq 1}))$ (see Lemma 1).

Lemma 4.1 *Let G be $\mathbb{P}_{\omega_2}$-generic and let $g = \mathcal{R}(\vec{z})$, $\vec{z} = (z_n)_{n\in\omega}$. Then $g \in O$ if and only if for every countable suitable model $\mathcal{M}$ such that $g \in \mathcal{M}$, there is $\bar{\alpha} < \omega_2^{\mathcal{M}}$ such that for all $n \in \omega$ the set $S_{\alpha+\omega\cdot n+m}$ is non-stationary in $(L[z_{n+1}])^{\mathcal{M}}$ for $m \in \Delta(z_n)$.*

Proof Suppose $g \in O$. Then $g = g_\alpha = \mathcal{R}(\vec{u}_\alpha)$ for some α. Let $\mathcal{M}$ be a countable suitable model such that $g \in \mathcal{M}$. By definition of the function $\bar{r}$ we have that $F^0(u_0^\alpha)$ and $F^1((u_n^\alpha))_{n\geq 1}$ are elements of $\mathcal{M}$. Since F^0, F^1 are nicely definable, $\vec{u}_\alpha \in \mathcal{M}$, and so $Y_{\alpha+\omega\cdot n} \cap \omega_1^{\mathcal{M}} \in \mathcal{M}$ for all n. Thus $X_{\alpha+\omega\cdot n} \cap \omega_1^{\mathcal{M}}$ is also an element of $\mathcal{M}$. By definition of $\mathcal{L}(\phi^n_{\alpha+\omega\cdot n})$, the set $X_{\alpha+\omega\cdot n} \cap \omega_1^{\mathcal{M}}$ codes a tuple $\langle W_j^n \rangle_{j\leq n}$ of subsets of ω_1 such that W_n^n is the L-least code of an ordinal α_n^n in ω_2 and for $j < n$ the set W_j^n is the L-least code for the largest limit ordinal α_j^n below α_{j+1}^n. It remains to observe that $W_j^n = W_j^m$ for $j \leq n < m$ and so α_0^n does not depend on n. But then $\bar{\alpha} = \alpha_0^n$ is the desired ordinal.

Suppose that for every countable suitable model $\mathcal{M}$ such that $g \in \mathcal{M}$, there is $\bar{\alpha} < \omega_2^{\mathcal{M}}$ with the desired properties. By the Löwenheim-Skolem theorem, the same holds in $\mathbb{H}_\Theta^{\mathbb{P}_{\omega_2}}$ for some large Θ. Therefore there is $\alpha < \omega_2^{\mathcal{M}}$ such that for all $n \in \omega$, the set $S_{\alpha+\omega\cdot n+m}$ is non-stationary iff $m \in \Delta(z_n)$. Since there is no accidental stationary kill, $z_n = u_n^\alpha$ for all n, which implies that $g = \mathcal{R}(\vec{u}_\alpha) = g_\alpha \in O$. □

Therefore O indeed has a Π^1_2-definition. We will show that O is maximal in $p_c(2^\omega)$. Suppose in $L^{\mathbb{P}_{\omega_3}}$ there is a code f of a measure orthogonal to every measure in the family $\bar{O} = \{\mu_g : g \in O\}$. Choose α minimal in $Lim'(\omega_2)$ such that $f \in L[G_\alpha]$ and let $x = (F^0_\alpha)^{-1}(f)$. Since $F^{-1}(\sigma_x^\alpha)$ is unbounded, there is $\alpha' \geq \alpha$ in $Lim'(\omega_2)$ such that $F(\alpha') = \sigma_x^\alpha (= \sigma_x^{\alpha'})$. But then $g_{\alpha'}$ is a code of a measure equivalent to μ_f, which is a contradiction. To obtain a Π^1_2-definable m.o. family in $L^{\mathbb{P}_{\omega_3}}$, consider the union of $\bar{O}$ with the set of all point measures. Just as in [1] one can show that $<$ is indeed a Δ^1_3-definable wellorder of the reals.

Since for every real $a \in L^{\mathbb{P}_{\omega_3}}$ there is a random real over L, by Proposition 1 in $L^{\mathbb{P}_{\omega_3}}$ there are no $\mathbf{\Sigma^1_2}$ m.o. families. The dominating number $\mathfrak{d}$ remains ω_1 in $L^{\mathbb{P}_{\omega_3}}$, since the countable support iteration of S-proper ${}^\omega\omega$-bounding posets is ${}^\omega\omega$-bounding (see [1, Lemma 18] or [5]). This completes our proof of Theorem 2.

We conclude with some open questions.

Remark 4.2 In [3] the following question was raised:

Question 1 If there is a Π^1_1 m.o. family, are all reals constructible?

This is to our knowledge still unsolved. Törnquist has recently shown that the existence of a Σ^1_2 m.o. family implies the existence of a Π^1_1 m.o. family, and that the existence of Σ^1_2 mad family implies the existence of a Π^1_1 mad family.

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