

Noetherian varieties in definably complete structures

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Abstract We prove that the zero-set of a C^∞ function belonging to a noetherian differential ring M can be written as a finite union of C^∞ manifolds which are definable by functions from the same ring. These manifolds can be taken to be connected under the additional assumption that every zero-dimensional regular zero-set of functions in M consists of finitely many points. These results hold not only for C^∞ functions over the reals, but more generally for definable C^∞ functions in a definably complete expansion of an ordered field. The class of definably complete expansions of ordered fields, whose basic properties are discussed in this paper, expands the class of real closed fields and includes o-minimal expansions of ordered fields. Finally, we provide examples of noetherian differential rings of C^∞ functions over the reals, containing non-analytic functions.

Keywords Noetherian varieties · Definable completeness · o-Minimality · Quasi-analytic functions

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1 Introduction

Definable completeness is a weak (first order expressible) version of Dedekind completeness.

Definition 1 Fix a language $L = \{+, -, \cdot, <, 0, \dots\}$ which is an expansion of the language of ordered rings. A *definably complete structure* \mathbb{K} (in the language L) is an

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L -expansion of an ordered field, such that every *definable* subset of the domain of \mathbb{K} which is bounded from above, has a least upper bound.

Let L be a language as in Definition 1. An L -expansion of an ordered field is *o-minimal* if every definable subset of the domain is a finite union of open intervals and points. Hence, every o-minimal L -expansion of an ordered field is definably complete. On the other hand, every definably complete structure expands a real closed field (see Theorem 2).

Definably complete structures have been studied mainly by C. Miller, some results can be found in [9].

Every L -expansion of the real ordered field \mathbb{R} is clearly definably complete, by Dedekind completeness. In particular, not every definably complete structure is o-minimal.

The notion of definable completeness is thus weaker than that of o-minimality (which has been extensively studied, see for example [1]). However, it has the advantage of being first order expressible. Unlike the o-minimal case, the class of all definably complete structures in a given language L is recursively axiomatizable. In particular, if \mathcal{R} is elementary equivalent to an expansion of the real ordered field, then \mathcal{R} is definably complete.

The aim of this paper is to lay the common groundwork for an axiomatic analysis of expansions of the real ordered field with some C^∞ functions, such as power functions, the exponential function, the sine function, quasi-analytic functions. The goal is to understand the geometric behaviour of sets definable in models of recursively axiomatized fragments of the theories of these structures, and hence to contribute to the solution of decidability questions related to these structures.

The main result of this paper is Theorem 33, which states that, in a definably complete structure \mathbb{K} , the zero-set of a C^∞ definable function belonging to a noetherian differential ring, can be written as a finite union of C^∞ \mathbb{K} -manifolds, which are definable by functions from the same ring.

Notice that we do not assume geometric finiteness (o-minimality, Pfaffianity, . . .), hence this decomposition theorem holds for a wide class of functions over definably complete structures. Over the real numbers, we find analytic but non-tame examples like $\sin(x)$, and in Sect. 7 we exhibit some C^∞ but non-analytic examples.

Results of a similar flavour have been obtained by Gabrielov [4] in the context of real analytic functions restricted to a compact ball.

This result is a first step in the analysis of quantifier-free definable sets in an expansion of the real field with noetherian functions.

We apply our decomposition theorem to prove a Khovanskii-type finiteness result (see Theorem 45): given a noetherian differential ring M of functions, if every zero-dimensional regular zero-set of functions in M consists of finitely many points, then the zero-set of any function in M has finitely many connected components.

Finally, we refine the candidate for a complete recursive axiomatization of the real exponential field \mathbb{R}_{\exp} proposed by Macintyre and Wilkie in [8]. We note that the axiom schemes $A3$, $A4$, T_{NA} , T_H in [8] follow from results 8, 14, 7 and 5, respectively.

The methods used throughout this paper are mainly due to Wilkie; here we generalize some results appearing in [8, 13].

2 Basic results

We fix, for the rest of this paper, a language L , which expands the language of ordered rings, and a definably complete L -structure \mathbb{K} . We equip \mathbb{K} with the interval topology, and any power \mathbb{K}^n with the product topology.

We give the usual ε, δ -definition of continuous function (where ε and δ are elements of \mathbb{K}). Limits are well defined, since the topology is Hausdorff.

The following classical results hold true in definably complete structures (the missing proofs are easy and can be found in [12]).

Theorem 2 (Intermediate value) *Let $a, b \in \mathbb{K}$ and $f: [a, b] \rightarrow \mathbb{K}$ be a continuous definable function such that $f(a) < 0$ and $f(b) > 0$. Then there exists $c \in (a, b)$ such that $f(c) = 0$.*

In particular, every definably complete structure is a real closed field.

Corollary 3 (Intermediate value property) *Let $a, b \in \mathbb{K}$ and $f: [a, b] \rightarrow \mathbb{K}$ be a continuous definable function. Then f takes all values in \mathbb{K} between $\inf f$ and $\sup f$ (which exist, possibly $\pm\infty$, by definable completeness).*

Theorem 4 (Weierstrass property) *Let $a, b \in \mathbb{K}$ and $f: [a, b] \rightarrow \mathbb{K}$ be a continuous definable function. Then f achieves maximum and minimum on $[a, b]$.*

A definable function $f: \mathbb{K} \rightarrow \mathbb{K}$ is differentiable at $x \in \mathbb{K}$ if the limit $\lim_{y \rightarrow x} \frac{f(y) - f(x)}{|y - x|}$ exists. Note that the derivative of a definable function (if it exists) is again a definable function. Analogously, if $f: \mathbb{K}^n \rightarrow \mathbb{K}$, we define the partial derivatives. As usual, we say that a definable function f is C^1 if it is differentiable, with continuous first derivatives. C^n and C^∞ are similarly defined. It is easy to see that derivatives satisfy the usual elementary properties, in particular the Chain Rule.

Theorem 5 (Taylor’s theorem)

- Let $F: \mathbb{K}^n \rightarrow \mathbb{K}$ be a definable C^N map and let $\bar{x}_0, \bar{x} \in \mathbb{K}^n$. Then there exists $\bar{\xi}$, lying on the line segment joining \bar{x}_0 and \bar{x} , such that

$$\begin{aligned}
 F(\bar{x}) &= F(\bar{x}_0) + F'(\bar{x}_0)[\bar{x} - \bar{x}_0] + \frac{F''(\bar{x}_0)}{2}[\bar{x} - \bar{x}_0, \bar{x} - \bar{x}_0] \\
 &+ \dots + \frac{F^{(N)}(\bar{\xi})}{N!}[\bar{x} - \bar{x}_0, \dots, \bar{x} - \bar{x}_0]
 \end{aligned}$$

- Let $F: \mathbb{K}^n \rightarrow \mathbb{K}^n$ be a definable C^{N+1} map and let $\bar{x}_0, \bar{x} \in \mathbb{K}^n$ such that $|\bar{x} - \bar{x}_0| < r$. Then

$$\begin{aligned}
 &\left| F(\bar{x}) - F(\bar{x}_0) - F'(\bar{x}_0)[\bar{x} - \bar{x}_0] - \dots - \frac{F^{(N)}(\bar{x}_0)}{N!}[\bar{x} - \bar{x}_0, \dots, \bar{x} - \bar{x}_0] \right| \\
 &\leq \sup_{\bar{y} \in B(\bar{x}_0, r)} \left| \frac{F^{(N+1)}(\bar{y})}{(N+1)!}[\bar{x} - \bar{x}_0, \dots, \bar{x} - \bar{x}_0] \right|
 \end{aligned}$$

Corollary 6 (Increasing functions and the sign of the derivative) *Let $a, b \in \mathbb{K}$ and $f: (a, b) \rightarrow \mathbb{K}$ be a C^1 definable function. If for all $x \in (a, b)$ we have $f'(x) > 0$, then f is strictly increasing on (a, b) .*

Let $F: \mathbb{K}^n \rightarrow \mathbb{K}^n$ be a C^2 definable map. Suppose we are given some point $\bar{x}_0 \in \mathbb{K}^n$ such that $|F(\bar{x}_0)|$ is small, $|F'(\bar{x}_0)|$ is bounded away from zero, and $|F'(\bar{x})|$ and $|F''(\bar{x})|$ are not too large on a suitable neighbourhood of \bar{x}_0 . Then F has a zero, moreover a nonsingular one, lying near to \bar{x}_0 . More precisely,

Theorem 7 (Newton’s method) *Let $a_0, a_1, a_2 \geq 1$. Then there exist $m, r \in \mathbb{K}^+$ (which can be written as rational functions of n, a_0, a_1, a_2) such that, $\forall \bar{x}_0 \in \mathbb{K}^n$,*

- If** $|F(\bar{x}_0)| < m$ and $\forall \bar{y} \in B(\bar{x}_0, r) |F'(\bar{y})^{-1}| < a_0$ and $|F'(\bar{y})| < a_1$ and $|F''(\bar{y})| < a_2$,
- Then** $\exists \bar{x} F(\bar{x}) = 0$ and $\bar{x} \in B(\bar{x}_0, r)$.

The proof is based on a repeated use of Taylor’s theorem.

Proof Let $r = (2n^3 a_0^2 a_1 a_2)^{-1}$ and $m = (4n^3 a_0^3 a_1 a_2)^{-1}$.

Let $\bar{x} \in B(\bar{x}_0, r)$ be such that $|F(\bar{x})| = \min\{|F(\bar{u})| : \bar{u} \in B(\bar{x}_0, r)\}$ (the existence of such a point \bar{x} follows from the fact that the function $\bar{u} \mapsto |F(\bar{u})|$ is continuous definable). We claim $F(\bar{x}) = 0$. Let

$$\bar{y} = \bar{x} - F'(\bar{x})^{-1} \cdot F(\bar{x}). \tag{1}$$

Equivalently, $F(\bar{x}) = F'(\bar{x})[\bar{x} - \bar{y}]$. It is sufficient to show that:

- (i) $\bar{y} \in B(\bar{x}_0, r)$;
- (ii) $|F(\bar{y})| \leq \frac{1}{2}|F(\bar{x})|$.

Proof of (i): By Taylor’s formula,

$$|F(\bar{x}_0) - F(\bar{x}) - F'(\bar{x})[\bar{x}_0 - \bar{x}]| \leq \sup \left| \frac{F''(\bar{z})}{2} [\bar{x}_0 - \bar{x}, \bar{x}_0 - \bar{x}] \right|, \tag{2}$$

where $\bar{y} \in B(\bar{x}_0, r)$. Hence,

$$|F(\bar{x}_0) - F(\bar{x}) - F'(\bar{x})[\bar{x}_0 - \bar{x}]| \leq \frac{a_2}{2} n^2 |\bar{x}_0 - \bar{x}|^2. \tag{3}$$

Now, using (1),

$$|F(\bar{x}_0) - F'(\bar{x})[\bar{x}_0 - \bar{y}]| \leq \frac{a_2}{2} n^2 |\bar{x}_0 - \bar{x}|^2. \tag{4}$$

Hence $|\bar{x}_0 - \bar{y}| \leq |F'(\bar{x})^{-1}| (|F(\bar{x}_0)| + \frac{a_2}{2} n^2 |\bar{x}_0 - \bar{x}|^2) \leq a_0(m + \frac{a_2}{2} n^2 r^2) \leq r$ (the last inequality can be easily checked by substituting the values of r, m). Therefore $\bar{y} \in B(\bar{x}_0, r)$.

Proof of (ii): By Taylor’s formula and using (1), we get

$$|F(\bar{y})| \leq |F(\bar{x}) + F'(\bar{x})[\bar{y} - \bar{x}]| + \frac{a_2}{2} n^2 |\bar{y} - \bar{x}|^2 = 0 + \frac{a_2}{2} n^2 |\bar{y} - \bar{x}|^2 \tag{5}$$

Another use of Taylor’s theorem yields

$$|F(\bar{x})| \leq |F(\bar{x}_0)| + a_1 n |\bar{x} - \bar{x}_0|. \tag{6}$$

Hence, by (1),

$$|\bar{y} - \bar{x}|^2 \leq |F'(\bar{x})^{-1}|^2 |F(\bar{x})|^2 \leq a_0^2 (m + a_1 nr) |F(\bar{x})|. \tag{7}$$

Putting all together, $|F(\bar{y})| \leq \frac{a_2}{2} n^2 a_0^2 (m + a_1 nr) |F(\bar{x})| \leq \frac{1}{2} |F(\bar{x})|$. □

Theorem 8 (Uniqueness theorem for systems of linear differential equations) *Let $a, b \in \mathbb{K} \cup \{\pm\infty\}$ and $F = (f_1, \dots, f_n) : (a, b) \rightarrow \mathbb{K}^n$ a C^∞ definable map. Let $A(t) = (a_{ij}(t))$ be an $n \times n$ matrix of C^∞ definable functions from (a, b) to \mathbb{K} ; suppose that*

$$\forall t \in (a, b) \quad F'(t) = A(t)F(t).$$

Then, either F is identically zero or else it never vanishes on (a, b) .

3 Varieties of C^∞ definable functions

In this section we develop some differential topology for the class of subsets of \mathbb{K}^n defined as zero-sets of definable smooth functions.

Definition 9 If $n, m \in \mathbb{N}$ and U is a definable open subset of \mathbb{K}^n , let $C^\infty(U, \mathbb{K}^m)$ be the ring of C^∞ definable maps from U to \mathbb{K}^m .

If $G \in C^\infty(U, \mathbb{K}^m)$, let the *variety of G* be the set $V(G) = \{\bar{a} \in U : G(\bar{a}) = \bar{0}\}$. If $G = (g_1, \dots, g_m)$, then $V(G) = V(g_1) \cap \dots \cap V(g_m)$; we will often write $V(g_1, \dots, g_m)$ instead of $V(G)$.

Remark 10 The variety of G is clearly a closed subset of U , for it is the preimage of a point under a continuous map.

Notation 11 If $n \geq m \in \mathbb{N}$, $G = (g_1, \dots, g_m) : \mathbb{K}^n \rightarrow \mathbb{K}^m$ is a C^∞ definable map and $\bar{a} \in \mathbb{K}^n$, we denote by $DG(\bar{a})$ the $m \times n$ matrix corresponding to the linear map $G'(a)$, with respect to the standard basis. The rows of $DG(\bar{a})$ are the vectors $\nabla g_1(\bar{a}), \dots, \nabla g_m(\bar{a})$. If $\bar{y} \subset \bar{x}$ is a sub-tuple of coordinates, then we denote by $D_{\bar{y}}G(\bar{a})$ the matrix of the partial derivatives $\frac{\partial g_i}{\partial y_j}(\bar{a})$ with respect to the variables in the tuple \bar{y} .

We will use many times in this work, some version of the Implicit Function Theorem. The statement is standard, but technical and we will find it useful to fix here a notation and to refer to this subsection whenever we use the theorem.

Definition 12 Let $n, m \in \mathbb{N}$, $U \subseteq \mathbb{K}^n$ open and definable and $G = (g_1, \dots, g_m) \in C^\infty(U, \mathbb{K}^m)$. Let $\bar{a} \in V(G)$ be a point such that the linear map $G'(\bar{a})$ is surjective.

Then we say that \bar{a} is a *regular point* of G . The set of regular zeroes of G (the *regular set* of G , for short) is denoted by $V^{\text{reg}}(G)$. So,

$$V^{\text{reg}}(G) := \{\bar{a} \in U : \bar{a} \in V(G) \text{ and } G'(\bar{a}) \text{ is onto}\}.$$

In other words, $V^{\text{reg}}(G)$ is the set of those $\bar{a} \in V(g_1, \dots, g_m)$ such that the vectors $\nabla g_1(\bar{a}), \dots, \nabla g_m(\bar{a})$ are \mathbb{K} -linearly independent (We denote by $\text{lin. span}\langle \nabla g_1(\bar{a}), \dots, \nabla g_m(\bar{a}) \rangle$ the \mathbb{K} -vector space generated by these vectors).

Notice that the definition of regular set depends crucially on the choice of G : as a set, $V(g_1) = V(g_1^2)$; but $V^{\text{reg}}(g_1) \neq V^{\text{reg}}(g_1^2)$. In fact $V^{\text{reg}}(g_1^2)$ is always empty.

We give now the notation which we will use for the Implicit Function Theorem and its corollaries.

Notation 13 Let $n \geq m \in \mathbb{N}$. We write $n = k + m$ and we fix the following set of coordinates:

$$\begin{aligned} \mathbb{K}^n &= \mathbb{K}^k \times \mathbb{K}^m \\ \bar{x} &= (\bar{u}, \bar{v}) \end{aligned}$$

Let $G = (g_1, \dots, g_m) \in C^\infty(\mathbb{K}^{k+m}, \mathbb{K}^m)$ and $\bar{x}_0 = (\bar{u}_0, \bar{v}_0) \in V(G)$ such that $D_{\bar{v}}G(\bar{x}_0)$ is non-singular.

Theorem 14 (Implicit function theorem) *There exist*

1. open definable subsets $O \subseteq \mathbb{K}^k$ and $W \subseteq \mathbb{K}^m$ such that $\bar{x}_0 \in O \times W$, and
2. a definable C^∞ map

$$Y : O \rightarrow W$$

such that $Y(\bar{u}_0) = \bar{v}_0$ and

$$\forall \bar{u} \in O \quad \forall \bar{v} \in W \quad G(\bar{u}, \bar{v}) = \bar{0} \Leftrightarrow \bar{v} = Y(\bar{u}).$$

Moreover, $D_{\bar{u}}Y(\bar{u})$ is everywhere non-singular and, if $J_{\bar{v}}G(\bar{x}) = \det D_{\bar{v}}G(\bar{x})$,

$$\forall \bar{x} \in U \quad D_{\bar{u}}Y(\bar{u}) = -J_{\bar{v}}G^{-1}(\bar{u}, Y(\bar{u})) \cdot D_{\bar{u}}G(\bar{u}, Y(\bar{u})).$$

Definition 15 The map

$$\begin{aligned} \phi : \mathbb{K}^k &\rightarrow V(G) \cap O \times W \\ \bar{u} &\mapsto (\bar{u}, Y(\bar{u})) \end{aligned}$$

is called a *local rectangular parametrization of $V(G)$ around \bar{x}_0* , and is a definable diffeomorphism, whose inverse is the restriction to $V(G) \cap (O \times W)$ of the projection $\pi : \mathbb{K}^n \rightarrow \mathbb{K}^k$ onto the first k coordinates.

The proof of the Implicit Function Theorem works as in the o-minimal case (see for example [1]). The only nontrivial fact in this setting, which is used in the proof, is the following.

Theorem 16 [9] *Let $A \subseteq \mathbb{K}^n$ be closed, bounded and definable, and let $f: A \rightarrow \mathbb{K}^m$ be a continuous definable map. Then $f(A)$ is closed, bounded and definable.*

We give now a list of the usual consequences of the Implicit Function Theorem.

Corollary 17 *There is a ring homomorphism (the restriction homomorphism)*

$$\begin{aligned} \widehat{\cdot}: C^\infty(O \times W, \mathbb{K}) &\rightarrow C^\infty(O, \mathbb{K}) \\ h &\mapsto \widehat{h}(\bar{u}) = h(\bar{u}, Y(\bar{u})) \end{aligned}$$

The kernel of $\widehat{\cdot}$ is the set $\{h \in C^\infty(O \times W, \mathbb{K}) : h \upharpoonright V(G) \cap (O \times W) \equiv 0\}$, hence

$$\widehat{C^\infty(O \times W, \mathbb{K})} \cong C^\infty(V(G) \cap (O \times W), \mathbb{K}).$$

Corollary 18 (Lagrange’s multipliers rule) *Let $h \in C^\infty(O \times W, \mathbb{K})$. A point $\bar{x} = (\bar{u}, Y(\bar{u})) \in V(G) \cap (O \times W)$ is a local extremum (maximum or minimum) of h on $V(G)$ if and only if $\nabla \widehat{h}(\bar{u}) = 0$. Moreover,*

$$\nabla \widehat{h}(\bar{u}) = 0 \Leftrightarrow \nabla h(\bar{u}, Y(\bar{u})) \in \text{lin. span}\langle \nabla g_1(\bar{u}, Y(\bar{u})), \dots, \nabla g_m(\bar{u}, Y(\bar{u})) \rangle.$$

Corollary 19 *Suppose $M \subset C^\infty(U \times W, \mathbb{K})$ is a noetherian ring closed under differentiation. Then so is $\widehat{M}[J_{\bar{v}}G^{-1}]$.*

Proof Notice that $\widehat{M}[J_{\bar{v}}G^{-1}]$ is a finite extension of a homomorphic image of a noetherian ring, hence it is noetherian; moreover, an easy calculation and Corollary 17 show that $\widehat{M}[J_{\bar{v}}G^{-1}]$ is also closed under differentiation. □

Definition 20 A definable subset W of \mathbb{K}^n is called a \mathbb{K} -manifold if there exists $k \leq n$ such that W is locally definably diffeomorphic to \mathbb{K}^k . More precisely, for every $\bar{a} \in W$ there exists an open definable neighbourhood $U_{\bar{a}} \subseteq \mathbb{K}^n$ of \bar{a} and a definable diffeomorphism $\Phi: U_{\bar{a}} \rightarrow \mathbb{K}^n$ such that $\Phi(U_{\bar{a}} \cap W) = \mathbb{K}^k \times \{\bar{0}\} \subset \mathbb{K}^n$. Notice that, for $\bar{a} \in W$, if $k', U'_{\bar{a}}, \Phi'$ are such that Φ' maps $U'_{\bar{a}} \cap W$ diffeomorphically onto $\mathbb{K}^{k'} \times \{\bar{0}\}$, then necessarily $k = k'$, because otherwise we would violate the chain rule (see the discussion below [3, Definition 3.1]). Hence, k is uniquely determined and will be called the \mathbb{K} -dimension of W (for short, $\text{dim}_{\mathbb{K}} W$).

Remark 21 It follows from the Implicit Function Theorem that, for $U \subseteq \mathbb{K}^n$ open and definable and $G \in C^\infty(U, \mathbb{K}^m)$, the set $V^{\text{reg}}(G)$ is either empty or it is a \mathbb{K} -manifold of \mathbb{K} -dimension $n - m$.

4 Noetherian differential rings

Definition 22 Let $n \in \mathbb{N}$ and $U \subseteq \mathbb{K}^n$ be a definably connected definable open set. A ring M with the following properties

- $M \subseteq C^\infty(U, \mathbb{K})$;
- M is noetherian;
- M is closed under partial differentiation;
- $M \supseteq \mathbb{Z}[x_1, \dots, x_n]$.

is called a *noetherian differential ring*.

Example 23 Let $\langle \mathbb{K}, +, -, \cdot, <, g_1, \dots, g_l, 0, 1, \dots \rangle$ be a definably complete structure such that $g_1, \dots, g_l \in C^\infty(\mathbb{K}^n, \mathbb{K})$ satisfy a system of differential equations, with polynomial coefficients:

$$\frac{\partial g_i}{\partial x_j}(\bar{x}) = q_{ij}(\bar{x}, g_1, \dots, g_l) \quad (i = 1, \dots, l; j = 1, \dots, n),$$

with $q_{ij} \in \mathbb{K}[\bar{x}, y_1, \dots, y_l]$.

Examples of such structures over the real numbers are: $\overline{\mathbb{R}} := \langle \mathbb{R}, +, -, \cdot, <, 0, 1, \dots \rangle, \langle \overline{\mathbb{R}}, \exp \rangle, \langle \overline{\mathbb{R}}, \tan \rangle, \langle \overline{\mathbb{R}}, \sin, \cos \rangle$ (notice that this last is a non o-minimal example).

Then,

$$\mathbb{F}[x_1, \dots, x_n, g_1, \dots, g_l],$$

where \mathbb{F} is a subfield of \mathbb{K} , is a noetherian differential subring of $C^\infty(\mathbb{K}^n, \mathbb{K})$.

Remark 24 In the given examples, if \mathbb{K} is a structure based on \mathbb{R} , then the functions g_1, \dots, g_l are not only C^∞ , but even analytic (by Cauchy–Kowalesky theorem, see for example [5]). On the other hand, if M is a noetherian differential ring which is *not* a finitely generated algebra, then it does not necessarily follow that M consists of real analytic functions. A non-analytic example will be exhibited in Section 7.

We now fix a definably connected definable open set $U \subseteq \mathbb{K}^n$ and a noetherian differential ring $M \subseteq C^\infty(U, \mathbb{K})$, and we study the properties of the zero-sets of functions in M . The methods are those used in [13]. The following result shows that the functions in M have a “quasi-analytic” behaviour.

Lemma 25 (Lack of flat functions) *Let $I \subseteq M$ be an ideal closed under differentiation; then either $V(I) = \emptyset$ or $V(I) = U$.*

Proof Since M is noetherian, I is finitely generated, say $I = \langle g_1, \dots, g_s \rangle$, and hence $V(I) = V(g_1, \dots, g_s)$ is a closed definable subset. If $V(I) \neq \emptyset$, since U is definably connected, all we need to show is that $V(I)$ is open.

Suppose for a contradiction that this is not the case. Then there exists $\bar{x} \in V(I)$ which is not an interior point, that is given an arbitrary open box neighbourhood B of \bar{x}_0 , there exists a point $\bar{y}_0 \in B$ which is not in $V(I)$. Without loss of generality, we may assume that \bar{x}_0, \bar{y}_0 differ in exactly one coordinate, say, the first one: $\bar{x}_0 = (s, p_2, \dots, p_n), \bar{y}_0 = (t, p_2, \dots, p_n)$ and $s \neq t$.

Recall that $\{g_1, \dots, g_s\}$ is a set of generators for I . Since I is closed under differentiation, it follows in particular that the derivatives with respect to the first coordinate

$\partial g_1/\partial x_1(\bar{x}), \dots, \partial g_s/\partial x_1(\bar{x})$ all belong to I , hence there exist functions $a_{ij}(\bar{x}) \in M$ such that

$$\forall \bar{x}, \forall i = 1, \dots, s \quad \frac{\partial g_i}{\partial x_1}(\bar{x}) = \sum_{j=1}^s a_{ij}(\bar{x})g_j(\bar{x}).$$

Now, consider the restrictions $f_i(x_1) = g_i(x_1, p_2, \dots, p_n)$ of the functions g_1, \dots, g_s to the line $L = \{\bar{x} \in U : x_2 = p_2 \wedge \dots \wedge x_n = p_n\}$, and define $F(x_1) = (f_1(x_1), \dots, f_s(x_1))$. We have

$$F'(x_1) = A(x_1)F(x_1),$$

where $A(x_1)$ is the $s \times s$ matrix whose entries are the functions $a_{ij}(x_1, p_2, \dots, p_n)$.

It follows from the uniqueness theorem for linear differential equations 8 that either $F \equiv 0$ or else has no zeros. But this leads to a contradiction, since $F(t) \neq 0$ and $F(s) = 0$. □

Corollary 26 *Let $G = (g_1, \dots, g_m) \in M^m$ and $\bar{x}_0 \in V^{\text{reg}}(G)$. Then either there exists $h \in M$ such that $\bar{x}_0 \in V^{\text{reg}}(G, h)$ or for all $h \in M$, if $h(\bar{x}_0) = 0$, then h vanishes on a definable neighbourhood of \bar{x}_0 in $V^{\text{reg}}(G)$.*

Proof We refer to the notation of the Implicit Function Theorem 14, so $\bar{x}_0 = (\bar{u}_0, \bar{v}_0) \in \mathbb{K}^k \times \mathbb{K}^m$. Up to some rearrangement of the variables, we may assume that $D_{\bar{v}}G(\bar{x}_0)$ is non-singular and apply the Implicit Function Theorem in a suitable neighbourhood $O \times W$ of \bar{x}_0 . Suppose that there is no $h \in M$ such that $\bar{x}_0 \in V^{\text{reg}}(G, h)$ and let $h \in M$ be such that $h(\bar{x}_0) = 0$. Then $\nabla h(\bar{x}_0)$ belongs to $\text{lin. span}\langle \nabla g_1(\bar{x}_0), \dots, \nabla g_m(\bar{x}_0) \rangle$. This implies, by Lagrange’s multiplier rule 18, that $\nabla \widehat{h}(\bar{u}_0) = 0$.

Consider the ideal $\widehat{I} = \{\widehat{g} \in \widehat{M}[\widehat{J}_{\bar{v}}G^{-1}] : \widehat{g}(\bar{u}) = 0\}$; what we have shown is that if $\widehat{h} \in \widehat{I}$, then its first derivatives $\partial \widehat{h}/\partial u_i$ belong to \widehat{I} ; thus \widehat{I} is closed under differentiation. Since $V(\widehat{I}) \neq \emptyset$, it follows from Lemma 25 and the subsequent remark, that $V(\widehat{I}) = O$. This means that h vanishes on $V^{\text{reg}}(G) \cap (O \times W)$. □

5 Decomposition of noetherian varieties

We fix, for the rest of the section, a noetherian differential ring $M \subset C^\infty(\mathbb{K}^n, \mathbb{K})$. The zero-sets of functions belonging to M are called M -varieties. We prove the main result of this paper, namely that every M -variety can be decomposed into finitely many \mathbb{K} -manifolds of a certain form.

Definition 27 Let A be a definable set; we say that S is a *definable clopen* of A if $S \subseteq A$ is a definable subset which is both open and closed in A . Clearly, the collection of all definable clopen of A is a boolean algebra $\mathcal{B}(A)$ of sets.

Definition 28 If $G \in M^m$ and S is a clopen definable subset of $V^{\text{reg}}(G)$, then S is called a *regular component*. Notice that S is a \mathbb{K} -manifold of the same \mathbb{K} -dimension as $V^{\text{reg}}(G)$, that is $n - m$.

Lemma 29 *Let $0 \neq f \in M$ and $V(f) \subset \mathbb{K}^n$ be a nonempty M -variety; then for all $\bar{x} \in V(f)$ there exists $g \in M$ such that $\bar{x} \in V^{\text{reg}}(g)$, that is $g(\bar{x}) = 0 \wedge \nabla g(\bar{x}) \neq 0$.*

Proof Take $\bar{x} \in V(f)$ and consider f together with all its partial derivatives, evaluated in \bar{x} . We claim that there exist a multi-index $\alpha = (\alpha_1, \dots, \alpha_n)$ and $i_0 \in \{1, \dots, n\}$ such that, if we put $\partial^\alpha f := \frac{\partial^{\alpha_1 + \dots + \alpha_n} f}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}}$, then $\partial^\alpha f(\bar{x}) = 0$ and $\frac{\partial \partial^\alpha f}{\partial x_{i_0}}(\bar{x}) \neq 0$, so that we can define $g := \partial^\alpha f$. Suppose, on the contrary, that f as well as all its derivatives $\partial^\alpha f$ vanishes in \bar{x} and let I be the ideal of M generated by f and all its derivatives. Notice that $V(I) \neq \emptyset$, since $\bar{x} \in V(I)$; M is noetherian, so I is finitely generated. Moreover, I is closed under differentiation, since each member of I can be written as a linear combination (with coefficients in M) of a finite number of derivatives of f . Then, Lemma 25 implies that $V(I)$, and hence $V(f)$, coincides with \mathbb{K}^n , which contradicts the fact that $0 \neq f$. □

Remark 30 The above argument shows also that, if $0 \neq f \in M$, then $V(f)$ has empty interior. In fact, if $V(f)$ has interior around a point \bar{x} , then \bar{x} is necessarily not a regular zero of f (otherwise, by the Implicit Function Theorem 14, $V(f)$ would be locally diffeomorphic to \mathbb{K}^{n-1} around \bar{x}). For the same reason, \bar{x} is not a regular zero of any of the derivatives of f , hence all the derivatives of any order of f vanish in \bar{x} . But then, as in the proof above, $V(f)$ must be \mathbb{K}^n .

Definition 31 For every $\bar{x} \in \mathbb{K}^n$, we define the M -degree of \bar{x} , $\text{deg}_M(\bar{x})$, as the minimal dimension of a regular component containing \bar{x} . Equivalently,

$$\text{deg}_M(\bar{x}) = \min\{k \mid \exists G \in M^{n-k} \text{ such that } \bar{x} \in V^{\text{reg}}(G)\}.$$

Lemma 29 shows that every point belonging to a proper M -variety has M -degree at most $n - 1$.

Theorem 32 *Let $f \in M$ and $V(f)$ be a proper M -variety. Then, for every point \bar{x} in $V(f)$, there exist $k < n$ and $G \in M^{n-k}$ and a regular component S of $V^{\text{reg}}(G)$ such that $\bar{x} \in S \subseteq V(f)$. Moreover, S is definable with the same parameters used to define G and f .*

Proof Let $k = \text{deg}_M(\bar{x})$ and $G \in M^{n-k}$ such that $\bar{x} \in V^{\text{reg}}(G)$. We define

$$S := \text{the interior of the set } V^{\text{reg}}(G) \cap V(f) \text{ in } V^{\text{reg}}(G).$$

We claim that $\bar{x} \in S$. In fact, by the choice of $V^{\text{reg}}(G)$ as a regular set of minimal \mathbb{K} -dimension, by Corollary 26 it follows that every function $h \in M$ which vanishes in \bar{x} , also vanishes on an open definable neighbourhood B of \bar{x} in $V^{\text{reg}}(G)$. In particular, f vanishes on some B (depending on f). Hence \bar{x} has an open neighbourhood B contained in $V(f) \cap V^{\text{reg}}(G)$, that is \bar{x} is an interior point.

We now claim that S is a regular component. S is definable, nonempty and open in $V^{\text{reg}}(G)$, by definition. We must show that S is also closed in $V^{\text{reg}}(G)$. Take a

boundary point \bar{x}_0 of S in $V^{\text{reg}}(G)$ and consider (after permuting the variables, if necessary) the local parametrization given by the Implicit Function Theorem 14

$$\begin{aligned} \phi : \mathbb{K}^k &\rightarrow V(G) \cap (O \times W) \\ \bar{u} &\mapsto (\bar{u}, Y(\bar{u})). \end{aligned}$$

Setting, as usual, $\hat{f} = f \circ \phi$, we observe that $\phi^{-1}(S)$ is open in O and $\hat{f}(\phi^{-1}(S)) = 0$. Hence, all derivatives of any order of \hat{f} vanish on $\phi^{-1}(S)$. Since $\bar{u}_0 = \phi^{-1}(\bar{x}_0)$ belongs to the closure of $\phi^{-1}(S)$, it is also true, by continuity, that \hat{f} , and all its derivatives of any order, vanish in \bar{u}_0 . By Lemma 25 and the usual argument, $V(\hat{f}) = \mathbb{K}^k$. Hence, the open neighbourhood $O \times W$ of \bar{x}_0 is contained in $V(f) \cap V^{\text{reg}}(G)$, that implies $\bar{x}_0 \in S$. \square

Theorem 33 (Decomposition of M -varieties) *Let $f \in M$ and $V(f)$ be a proper M -variety. Then $V(f)$ can be written as a finite union of regular components:*

$$\exists k \in \mathbb{N}, \exists G_1, \dots, G_k \in \bigcup_{l=1}^n M^l, \exists S_i \in \mathcal{B}(V^{\text{reg}}(G_i)) \text{ so that } V(f) = S_1 \cup \dots \cup S_k.$$

Proof By compactness. More precisely, let \mathbb{F} be a $|\mathbb{K}|^+$ -saturated elementary superstructure of \mathbb{K} (see [11] for the existence of such an \mathbb{F}), so that \mathbb{F} realizes all types over \mathbb{K} . Let \tilde{M} be the set of those definable functions \tilde{g} such that $g \in M$ and \tilde{g} is the interpretation of g in \mathbb{F} (note that \tilde{g} is still a C^∞ function). Then \tilde{M} is still a noetherian differential ring, hence Theorem 32 holds for \tilde{M} -varieties. Consider the function \tilde{f} and the following set of formulas:

$$\begin{aligned} \Phi = \left\{ \phi_{\tilde{G}} := \bar{x} \in V(\tilde{f}) \wedge \tilde{S} = \text{int}_{V^{\text{reg}}(\tilde{G})}(V^{\text{reg}}(\tilde{G} \cap V(\tilde{f}))) \wedge (\bar{x} \in \tilde{S} \right. \\ \left. \rightarrow \tilde{S} \not\subseteq V(\tilde{f})) \mid \tilde{G} \in \bigcup_{i=1}^n \tilde{M}^i \right\}. \end{aligned}$$

If Φ were a consistent type, then it would be realized in \mathbb{F} . This means that there would exist an $\bar{x} \in \mathbb{F}$ such that for all $\tilde{G} \in \bigcup_{i=1}^n \tilde{M}^i$, $\bar{x} \in V(\tilde{f}) \wedge \tilde{S} = \text{int}_{V^{\text{reg}}(\tilde{G})}(V^{\text{reg}}(\tilde{G} \cap V(\tilde{f}))) \wedge (\bar{x} \in \tilde{S} \rightarrow \tilde{S} \not\subseteq V(\tilde{f}))$, which would contradict Theorem 32. Hence there exist $k \in \mathbb{N}$, $\tilde{G}_1, \dots, \tilde{G}_k \in \bigcup_{i=1}^n \tilde{M}^i$, such that the conjunction $\phi_{\tilde{G}_1} \wedge \dots \wedge \phi_{\tilde{G}_k}$ is not satisfiable in \mathbb{F} ; in other words the following holds in \mathbb{F} :

$$\forall \bar{x} (\bar{x} \in \tilde{S}_1 \cup \dots \cup \tilde{S}_k \wedge \tilde{S}_1 \cup \dots \cup \tilde{S}_k \subseteq V(\tilde{f})).$$

Therefore $V(\tilde{f}) = \tilde{S}_1 \cup \dots \cup \tilde{S}_k$.

Now, in \mathbb{K} the following holds: $V(f) = S_1 \cup \dots \cup S_k$, where $S_i := \tilde{S}_i \cap \mathbb{K}^n$ ($i = 1, \dots, k$) are clearly regular components in \mathbb{K} , hence the theorem is proved. \square

Remark 34 The decomposition which appears in Theorem 33 is clearly not unique, nor are unique the \mathbb{K} -dimensions of the regular components appearing in two different decompositions of the same variety. For example, the algebraic variety $V(x^2 - y^2) \subset \mathbb{R}^2$ can be decomposed as $V^{\text{reg}}(x^2 - y^2) \cup V^{\text{reg}}(x, y)$ or as $V^{\text{reg}}(x - y) \cup V^{\text{reg}}(x + y)$. In the first decomposition the first regular component has \mathbb{K} -dimension 1 and the second has \mathbb{K} -dimension 0, while in the second decomposition both regular components have \mathbb{K} -dimension 1. Moreover, in the first case the union is disjoint, and in the second case it is not.

On the other hand, the following holds:

Lemma 35 *Let $f \in M$. Then there exists a unique natural number m such that for every decomposition of $V(f)$ into regular components, as in Theorem 33, the maximal \mathbb{K} -dimension of the regular components appearing in the decomposition is m .*

Proof Let $V(f) = S_1 \cup \dots \cup S_k$ be a decomposition of $V(f)$ into regular components and suppose $\dim_{\mathbb{K}} S_i \leq \dim_{\mathbb{K}} S_1 = m$, for all $i = 2, \dots, k$. Clearly $V(f)$ does not contain an open subset which is diffeomorphic to \mathbb{K}^l , for $l > m$, because otherwise such a subset would be obtained as a finite union of manifolds of \mathbb{K} -dimension $\leq m$, which, as in the classical case, is not possible. On the other hand, $V(f)$ does contain an open subset which is diffeomorphic to \mathbb{K}^m , because so does S_1 . Hence, every decomposition of $V(f)$ must contain a component of \mathbb{K} -dimension m , and can not contain components of bigger \mathbb{K} -dimension. □

Definition 36 The *dimension* of an M -variety $V(f)$ is defined as

$$\dim V(f) := \max\{\dim_{\mathbb{K}} S_i \mid i = 1, \dots, k \text{ and } V(f) = S_1 \cup \dots \cup S_k\},$$

where $V(f) = S_1 \cup \dots \cup S_k$ is any decomposition given by Theorem 33.

Remark 37 Notice that, for *regular* varieties, the notions of dimension and \mathbb{K} -dimension coincide.

Lemma 38 *Let $g_1, \dots, g_m \in M$ and $\bar{x} \in V^{\text{reg}}(g_1, \dots, g_m) \subset \mathbb{K}^n$. If $\deg_M(\bar{x}) < n - m$, then there exists $f \in M$ such that $\bar{x} \in V^{\text{reg}}(g_1, \dots, g_m, f)$.*

Proof Since $\deg_M(\bar{x}) < n - m$, there exist $f_1, \dots, f_{m+1} \in M$ so that $\bar{x} \in V^{\text{reg}}(f_1, \dots, f_{m+1})$. We claim that there exists $i \in \{1, \dots, m+1\}$ so that $\bar{x} \in V^{\text{reg}}(g_1, \dots, g_m, f_i)$, because otherwise the (linearly independent) vectors $\nabla f_1(\bar{x}), \dots, \nabla f_{m+1}(\bar{x})$ would all lie in the m -dimensional vector space generated by $\nabla g_1(\bar{x}), \dots, \nabla g_m(\bar{x})$, which is impossible. □

Proposition 39 *Let $V(f) \subset \mathbb{K}^n$ be an M -variety. Let \mathbb{F} be any $|\mathbb{K}|^+$ -saturated elementary superstructure of \mathbb{K} and let \tilde{f} be the interpretation of f in \mathbb{F} (as in the proof of 33). Then,*

$$\dim V(f) = \max\{\deg_M(\bar{x}) \mid \bar{x} \in V(\tilde{f})\}.$$

Proof Let $V(\tilde{f}) = \tilde{S}_1 \cup \dots \cup \tilde{S}_l$ be a decomposition of $V(\tilde{f})$ into regular components, and let $S_i = \mathbb{K}^n \cap \tilde{S}_i$. Then $V(f) = S_1 \cup \dots \cup S_l$, hence $\dim V(\tilde{f}) = \dim V(f)$. Let $\bar{x} \in V(\tilde{f})$. Then $\bar{x} \in \tilde{S}_i$ for some i , hence $\deg_M(\bar{x}) \leq \dim \tilde{S}_i \leq \dim V(\tilde{f})$. So $\dim V(f) \geq \max\{\deg_M(\bar{x}) \mid \bar{x} \in V(\tilde{f})\}$.

Now we prove that there exists $\bar{x} \in V(\tilde{f})$ with $\deg_M(\bar{x}) = \dim V(f)$. Let $\tilde{S}_1 \in \mathcal{B}(V^{\text{reg}}(\tilde{g}_1, \dots, \tilde{g}_k))$ be a component of maximal dimension. Consider the set of formulas

$$\Phi = \{\bar{x} \in \tilde{S}_1 \wedge \bar{x} \notin V^{\text{reg}}(\tilde{g}_1, \dots, \tilde{g}_k, \tilde{h}) \mid h \in M\}.$$

Φ is clearly finitely satisfiable in \mathbb{F} , because no finite union of regular sets of dimension $n - k - 1$ can cover the whole of \tilde{S}_1 , which has dimension $n - k$. By saturation, there exists $\bar{x} \in \mathbb{F}^n$ which satisfies all formulas in Φ . By Lemma 38, then, $\deg_M(\bar{x}) = \dim V(f)$. □

6 Khovanskii rings

In this section we give an application of the Decomposition Theorem 33. We consider a class of noetherian differential rings, called *Khovanskii rings*, with the property that 0-dimensional regular sets are finite. An example of such a ring is $M = \mathbb{R}[\bar{x}, f_1, \dots, f_k]$, where the functions f_i form a Pfaffian chain, as proved by Khovanskii in [6]. In the same paper, it is proven that all M -varieties, where M is as above, have finitely many connected components. Here we prove, with a method which differs from the approach in [6], that if M is a Khovanskii ring in a definably complete structure, then all M -varieties have finitely many *definably connected components*.

This result, which generalizes and is proved with the same methods used in [8, Theorem 2.2], can be compared with [3, Theorem 1.7], which is formulated in the setting of Rolle leaves.

Definition 40 (Khovanskii rings) Let $n \in \mathbb{N}$. A ring M with the following properties

1. $M \subseteq C^\infty(\mathbb{K}^n, \mathbb{K})$;
2. M is a noetherian differential ring;
3. $\forall g_1, \dots, g_n \in M \quad |V^{\text{reg}}(g_1, \dots, g_n)| < \infty$.

is called a *Khovanskii ring*.

A collection of rings $\{M_n \mid n \in \mathbb{N}\}$ such that

1. M_n is a ring of definable C^∞ functions from \mathbb{K}^n to \mathbb{K} ;
2. M_n is a Khovanskii ring;
3. $M_n \subset M_{n+1}$ (in the obvious sense);
4. M_n is closed under permutation of the variables.

is called a *collection of Khovanskii rings*.

A similar definition appears in [2].

Examples 41 Examples of Khovanskii rings over the real numbers are:

- The ring generated by a Pfaffian chain of functions (see [6]);

- Any noetherian differential ring of functions definable in an o-minimal expansion of the real field;
- The ring generated by the real functions $\exp(\frac{1}{1+x^2})$, $\sin(\frac{1}{1+x^2})$ and $\cos(\frac{1}{1+x^2})$ (see [7]).

Remark 42 Fix $n, m \in \mathbb{N}$, $m \leq n$. Let $M \subseteq C^\infty(\mathbb{K}^n, \mathbb{K})$ be a noetherian differential ring (not necessarily a Khovanskii ring) and let $F \in M^m$. Then the set of regular zeroes of F can be expressed as the projection of a finite union of *regular* varieties of dimension $\dim V^{\text{reg}}(F)$. To see this, let $E_1(\bar{x}), \dots, E_l(\bar{x})$ be the maximum rank minors of the matrix $DF(\bar{x})$. Now consider $V_i := V(F(\bar{x}), x_{n+1} \det E_i(\bar{x}) - 1)$. Then V_i is a regular sub-variety of \mathbb{K}^{n+1} and $\pi_{n+1}(\bigcup_{i=1}^l V_i) = V^{\text{reg}}(F)$ (where π_{n+1} is the projection onto the first n coordinates).

Notice that $\dim V_i = n + 1 - (m + 1) = n - m = \dim V^{\text{reg}}(F)$. Moreover, if $M = M_n$ belongs to a collection of Khovanskii rings, then the map $(F(\bar{x}), x_{n+1} \det E_i(\bar{x}) - 1)$ belongs to M_{n+1}^{n-k+1} .

Proposition 43 Fix $n, m \in \mathbb{N}$, $m \leq n - 1$. Let $M \subseteq C^\infty(\mathbb{K}^n, \mathbb{K})$ be a Khovanskii ring and $f_1, \dots, f_m \in M$ be such that $V(f_1, \dots, f_m) = V^{\text{reg}}(f_1, \dots, f_m)$. Then there exists a definable set G such that:

- $\emptyset \neq G \subset V(f_1, \dots, f_m)$;
- For every clopen definable subset S of $V(f_1, \dots, f_m)$, the intersection $S \cap G$ is not empty;
- $\forall \bar{x} \in G \exists h \in M (\bar{x} \in V^{\text{reg}}(f_1, \dots, f_m, h))$.

Proof For all $h \in M$, consider the matrix of partial derivatives $D(f_1, \dots, f_m, h)$. Let $\bar{x} \in V(f_1, \dots, f_m)$. Then, this matrix, if we evaluate all the entries in \bar{x} , has rank at least m , because the common zeroes of the functions f_1, \dots, f_m are all regular zeroes, by hypothesis. Let $H_i(\bar{x})$ ($i = 1, \dots, \binom{n}{m+1}$) be the minors of rank $m + 1$ of the matrix $D(f_1, \dots, f_m, h)$ evaluated in \bar{x} and define $h^*(\bar{x}) = \sum_{i=1}^l (\det H_i(\bar{x}))^2 \in M$. Then \bar{x} is a critical point of h on $V(f_1, \dots, f_m)$ if and only if $h^*(\bar{x}) = 0$. And (see 18),

$$h^*(\bar{x}) = 0 \Leftrightarrow \nabla h(\bar{x}) \in \text{lin. span}(\nabla f_1(\bar{x}), \dots, \nabla f_m(\bar{x})).$$

We take $n + 1$ points P_0, \dots, P_n in \mathbb{Z}^n such that the vectors $\mathbf{P}_0\mathbf{P}_1, \dots, \mathbf{P}_0\mathbf{P}_n$ are linearly independent over \mathbb{K} . For example, let us take $P_0 = \bar{0}$ and P_i to be the tuple with the i -th coordinate equal to 1 and the other coordinates equal to 0 (for $i = 1, \dots, n$). Now consider the following “distance” functions:

$$d_0(\bar{x}) = \sum_{j=1}^n x_j^2, \quad d_i(\bar{x}) = (x_i - 1)^2 + \sum_{j=1, j \neq i}^n x_j^2 \quad i = 1, \dots, n.$$

Clearly these functions belong to M .

For every $S \in \mathcal{B}(V(f_1, \dots, f_m))$, for every $i = 0, \dots, n$, consider the set $V_S(d_i^*) = S \cap V(d_i^*)$ of the critical points of the function d_i on S and let $\text{bd}_S V_S(d_i^*) = V_S(d_i^*) \setminus \text{int}_S(V_S(d_i^*))$ be the set of boundary points of $V_S(d_i^*)$ in S . Now define

$$G := \bigcup_{S \in \mathcal{B}(V(f_1, \dots, f_m))} \bigcup_{i=0, \dots, n} \text{bd}_S V_S(d_i^*).$$

Step 1. We first observe that G is definable and $G \subseteq V(f_1, \dots, f_m)$.

Step 2. Next, we note that for all $S \in \mathcal{B}(V(f_1, \dots, f_m))$, for every $i = 0, \dots, n$, the set S contains a point whose distance from P_i is minimal, that is $V_S(d_i^*)$ is nonempty. This follows from the fact that d_i increases on balls centered in P_i and of increasing radius, so Theorem 4 applies.

Step 3. Now we show that G meets every nonempty definable clopen of $V(f_1, \dots, f_m)$ (in particular, G is not empty). Equivalently, we show that for all $S \in \mathcal{B}(V(f_1, \dots, f_m)) \setminus \{\emptyset\}$, there exists $i \in \{0, \dots, n\}$ such that the set $V_S(d_i^*)$ is not open in S . Suppose for a contradiction that this is not the case. Then for all $i = 0, \dots, n$ the set $V_S(d_i^*)$, which is clearly closed and definable, is also open in $V(f_1, \dots, f_m)$, and hence it belongs to $\mathcal{B}(V(f_1, \dots, f_m))$. Now consider the boolean subalgebra \mathcal{A} of $\mathcal{B}(V(f_1, \dots, f_m))$ generated by $V_S(d_0^*), \dots, V_S(d_n^*)$. Since \mathcal{A} is finite, there is an atom, say, $C \in \mathcal{A}$. Let $C_i = C \cap V_S(d_i^*)$; by Step 1, C_i is nonempty for all $i = 0, \dots, n$, and hence $C_i = C$. But this implies that $\emptyset \neq C \subseteq V(d_0^*, \dots, d_n^*)$. But this is not possible, because the vectors $\nabla d_i(\bar{x})$ span \mathbb{K}^n at all points \bar{x} . If $\bar{x} \in V(d_0^*, \dots, d_n^*)$, then $\forall i = 0, \dots, n, \nabla d_i(\bar{x}) \in \text{lin. span}(\nabla f_1(\bar{x}), \dots, \nabla f_m(\bar{x}))$, which is absurd.

Step 4. We now show that $\forall \bar{x} \in G \exists h \in M \bar{x} \in V^{\text{reg}}(f_1, \dots, f_m, h)$. Suppose for a contradiction that there exists $\bar{x} \in G$ such that it is not possible to cut transversally $V(f_1, \dots, f_m)$ at \bar{x} by any $h \in M$. Now arguing as in the last paragraph of the proof of Theorem 32, we show that every $h \in M$ must vanish on a suitable neighbourhood of \bar{x} in $V(f_1, \dots, f_m)$. But by definition of G , every point \bar{x} of G is a boundary point of some $V_S(d_i^*)$, that is

$$\begin{aligned} \forall \bar{x} \in G \quad \exists S \in \mathcal{B}(V(f_1, \dots, f_m)) \quad \exists i \in \{0, \dots, s\} \\ d_i^*(\bar{x}) = 0 \quad \wedge \quad \forall r > 0 \exists \bar{y} \in S \cap B(\bar{x}, r) \quad d_i^*(\bar{y}) \neq 0, \end{aligned} \tag{8}$$

and this leads to a contradiction. □

Theorem 44 Fix $n, m \in \mathbb{N}, m \leq n - 1$. Let $M \subseteq C^\infty(\mathbb{K}^n, \mathbb{K})$ be a Khovanskii ring and $F \in M^m$ be such that $V(F) = V^{\text{reg}}(F)$. Then there exists a definable set G such that:

- $\emptyset \neq G \subset V(F)$;
- For every clopen definable subset S of $V(F)$, the intersection $S \cap G$ is not empty;
- $\exists l \in \mathbb{N}, \exists h_1, \dots, h_l \in M \quad G \subset V^{\text{reg}}(F, h_1) \cup \dots \cup V^{\text{reg}}(F, h_l)$.

Proof By compactness, using an argument similar to the one used in the proof of 33. More precisely, let \mathbb{F} be a $|\mathbb{K}|^+$ -saturated elementary superstructure of \mathbb{K} (see [11] for the existence of such an \mathbb{F}), so that \mathbb{F} realizes all types over \mathbb{K} . Let \tilde{M} be the set of those definable functions \tilde{g} such that $g \in M$ and \tilde{g} is the interpretation of g in \mathbb{F} (note

that \tilde{g} is still a C^∞ function). Then \tilde{M} is still a Khovanskii ring, hence Proposition 43 holds for \tilde{M} -varieties. Consider the map $\tilde{F} = (\tilde{f}_1, \dots, \tilde{f}_m)$ and the following set of formulas:

$$\Phi = \{\phi_{\tilde{h}} := (\bar{x} \in G \wedge \bar{x} \notin V^{\text{reg}}(\tilde{F}, \tilde{h})) \mid h \in M\}.$$

If Φ were a consistent type, then it would be realized in \mathbb{F} . This means that there would exist $\bar{x} \in G$ such that for all $h \in M$, $\bar{x} \notin V^{\text{reg}}(\tilde{F}, \tilde{h})$, which is not possible by Proposition 43. Hence there exist $h_1, \dots, h_l \in M$ such that the conjunction $\phi_{\tilde{h}_1} \wedge \dots \wedge \phi_{\tilde{h}_l}$ is not satisfiable; in other words the following holds in \mathbb{F} :

$$\forall \bar{x} \bar{x} \in G \rightarrow \bar{x} \in V^{\text{reg}}(\tilde{F}, \tilde{h}_1) \cup \dots \cup V^{\text{reg}}(\tilde{F}, \tilde{h}_l).$$

Pulled back to \mathbb{K} , this proves the theorem. □

Theorem 45 (Finiteness of $\mathcal{B}(V(F))$) *Let $\{M_n \mid n \in \mathbb{N}\}$ be a collection of Khovanskii rings. Then, for all $n, m \in \mathbb{N}$ and $F \in (M_n)^m$, the boolean algebra $\mathcal{B}(V(F))$ is finite.*

Proof By induction, using Propositions 44, 33 and Remark 42.

More precisely, we first prove by induction on $k = n - m$ that $\forall n \in \mathbb{N}, \mathcal{B}(V^{\text{reg}}(F))$ is finite.

The case $k = 0$ follows from the fact that M_n is a Khovanskii ring. Next, suppose the statement true for $n - m < k$ and consider $F \in M_n^{n-k}$. If $V(F) = V^{\text{reg}}(F)$, then there exist a definable set G and functions $h_1, \dots, h_l \in M_n$ as in Theorem 44.

By inductive hypothesis, $\mathcal{B}(V^{\text{reg}}(F, h_i))$ is finite, and hence so is $\mathcal{B}(V^{\text{reg}}(F))$, assuming $V(F) = V^{\text{reg}}(F)$.

If $V(F) \neq V^{\text{reg}}(F)$, then, by remark 42, $V^{\text{reg}}(F)$ is the projection of a finite union of regular varieties V_i still of dimension k , hence it follows from what we have just proved that $\mathcal{B}(V_i)$ is finite, and hence so is $\mathcal{B}(\pi(\bigcup V_i)) = \mathcal{B}(V^{\text{reg}}(F))$.

Finally, if $V(F)$ is any variety, not necessarily regular, then by Theorem 33 it follows that $V(F)$ is a finite union of clopen subsets of regular sets, hence, by what we have just proved, $\mathcal{B}(V(F))$ is finite. □

Remark 46 (Definably connected components) Since the boolean algebra $\mathcal{B}(V(F))$ is finite, then there is an atom. If A is an atom, then it is clearly a maximal definably connected subset, that is a definably connected component. Hence we have proved that $V(F)$ has a finite number of definably connected components.

7 A non-analytic example

We give an example of noetherian differential ring over the real numbers, which contains non-analytic functions. We need to give some definitions.

Let X is a closed and bounded interval $[a, b] \subset \mathbb{R}$. Let $C^\infty(X, \mathbb{R})$ be the ring of all real functions which are C^∞ on a neighbourhood of X .

Definition 47 A function $f \in C^\infty(X, \mathbb{R})$ is *flat at the point* $x_0 \in X$ if all derivatives of f vanish in x_0 , but there is no neighbourhood of x_0 where f assumes only the value zero. A function $f \in C^\infty(X, \mathbb{R})$ is *flat* if it is flat at some point $x_0 \in X$.

Definition 48 A ring $M \subseteq C^\infty(X, \mathbb{R})$ is *quasi-analytic* if for every $x_0 \in X$ the ring homomorphism

$$\begin{aligned} \eta: M &\rightarrow \mathbb{R}[[x]] \\ f &\mapsto \text{the Taylor expansion of } f \text{ at } x_0 \end{aligned}$$

is injective. Equivalently, M does not contain flat functions.

Remark 49 The arguments appearing in the proof of Lemma 25 show that if $M \subseteq C^\infty(X, \mathbb{R})$ is noetherian and closed under differentiation, then M is quasi-analytic.

Definition 50 A ring $N \subseteq C^\infty(X, \mathbb{R})$ is *closed under taking C^∞ -quotients* if the following holds: for all $f, g \in N$, if $h := \frac{f}{g}$ is C^∞ on X , then $h \in N$.

Given a ring $M \subseteq C^\infty(X, \mathbb{R})$, we denote by \overline{M} the *closure of M under C^∞ -quotients*, that is the smallest ring containing M and closed under taking C^∞ -quotients.

The following result is well known folklore.

Theorem 51 Let $M \subseteq C^\infty(X, \mathbb{R})$ be a quasi-analytic ring, which is closed under differentiation and contains the coordinate functions. Then its closure under C^∞ -quotients \overline{M} is closed under differentiation, quasi-analytic and noetherian.

Remark 52 The theorem holds true, with the same proof, if $X = \mathbb{R}$ and all functions in M have finitely many zeroes with finite multiplicities. In particular, if M consists of functions which are definable in some *o-minimal* expansion of the real field, then \overline{M} is noetherian.

Definition 53 Let $M \subset C^\infty([0, 1], \mathbb{R})$ be a quasi-analytic ring, closed under differentiation and containing the coordinate function x . Let $N := \{f \upharpoonright (0, 1) \mid f \in \overline{M}\}$ and $\mathcal{M} := N[x]$.

$\mathcal{M} \subset C^\infty(\mathbb{R}, \mathbb{R})$ is clearly noetherian and closed under differentiation. Moreover, if M contains a non-analytic function, then so does \mathcal{M} .

Example 54 A concrete example of the above construction can be obtained by taking as M a quasi-analytic Denjoy–Carleman class on $[0, 1]$ (according to the definition in [10]).

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