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On zeros of Martin-Löf random Brownian motion

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Abstract: We investigate the sample path properties of Martin-Löf random Brownian motion. We show (1) that many classical results which are known to hold almost surely hold for every Martin-Löf random Brownian path, (2) that the effective dimension of zeroes of a Martin-Löf random Brownian path must be at least 1/2, and conversely that every real with effective dimension greater than 1/2 must be a zero of some Martin-Löf random Brownian path, and (3) we will demonstrate a new proof that the solution to the Dirichlet problem in the plane is computable.

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1 Background and notation

1.1 Brownian motion

Heuristically, Brownian motion is the random continuous function resulting from the limit of discrete random walks as the time interval approaches zero. The paths of Brownian motion are considered typical with respect to *Wiener measure* on a function space, generally $C[0,1]$, $C[0,\infty)$, or $C[I,\mathbb{R}^n]$ for $I=[0,1]$ or $[0,\infty)$. The Martin-Löf random elements of a function space with respect to Wiener measure are known as Martin-Löf random Brownian motion. Fouché showed that the class of Martin-Löf random Brownian motion is the same as the class of complex oscillations, a class of functions defined by Asarin and Pokrovskii [2] and later investigated to a greater degree by Fouché [7, 8, 9], Davie and Fouché [5], Kjos-Hanssen and Nerode [16], and Szabados [17].

In this article, we continue the study of Martin-Löf random Brownian motion. We will demonstrate that many classical theorems which hold almost surely hold for every Martin-Löf random Brownian path, we will prove results toward a classification of

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the effective dimension of the zeroes of Martin-Löf random Brownian motion, and we will demonstrate a new proof that the solution to the Dirichlet problem in the plane is computable.

We use 2^ω to denote infinite binary strings, which we sometimes identify with reals on [0, 1]. We denote the space of continuous functions $f : [0,1] \to \mathbb{R}$ and $f : [0,\infty) \to \mathbb{R}$ by $C[0,1]$ and $C[0,\infty)$ respectively. For other cases, the space of continuous functions from a set X to a set Y will be denoted by $C(X,Y)$.

Standard (one dimensional) Brownian motion is a real-valued stochastic process $\{\mathscr{B}(t) : t \in I\}$ ($I = [0,1]$ or $I = [0,\infty)$) where the following hold. First, for any $t_0 < t_1 < \cdots < t_n$ the increments $\mathscr{B}(t_n) - \mathscr{B}(t_{n-1}), \mathscr{B}(t_{n-1}) - \mathscr{B}(t_{n-2}), \ldots, \mathscr{B}(t_2) - \mathscr{B}(t_1)$ are independent random variables. Second, for all $t \geq 0$ and $h > 0$ the increments $\mathscr{B}(t+h) - \mathscr{B}(t)$ are normally distributed with mean 0 and variance h. Third, $\mathscr{B}(0) = 0$ almost surely, and $\mathscr{B}$ is almost surely continuous. These requirements induce a unique measure on the space of real-valued continuous functions over I called Wiener measure, which we denote by $\mathbb{P}$. The values taken by the random variable $\mathscr{B}$ are called *sample paths*, or simply *paths*.

It is possible to define Brownian motion starting at any point x at time 0, rather than starting at the origin, in which case we will denote the corresponding measure by $\mathbb{P}_x$ (in other words, $\mathbb{P}_x(\mathscr{B} \in \mathcal{A}) = \mathbb{P}(x + \mathscr{B} \in \mathcal{A})$). When we wish to emphasize that we are talking about standard Brownian motion, we will use $\mathbb{P}_0$.

We assume that the reader is familiar with algorithmic randomness and Kolmogorov complexity for binary sequences. One can consult the two books by Downey and Hirschfeldt [6] and Nies [21] for a good overview of the subject. Furthermore, we assume some familiarity with Martin-Löf randomness for computable probability spaces. Gács' lecture notes [11] and the two papers [12, 13] by Hoyrup and Rojas are the standard references on the subject. Our main reference for the classical theory of Brownian motion is the recent book by Mörters and Peres [20].

1.2 Effective aspects of Brownian motion

In order to define Martin-Löf randomness for Brownian motion, one needs to make sure that the space of continuous functions $C[0,1]$ endowed with distance

$$d(f,g) = ||f - g||_\infty$$

and Wiener measure (denoted $\mathbb{P}$) is a computable probability space.

The computability of $(C[0,1],\mathbb{P})$ was proven by Fouché and Davie [5, 8] (see next subsection for more details). One can take for a dense set of points the piecewise linear functions which interpolate between finitely many points of rational coordinates. For p such a function and $r>0$ a rational number, the $\mathbb{P}$-measure of

$$\{f \mid ||f-p||_\infty < r\}$$

is computable uniformly in a code for p.

Therefore, it is possible to define Martin-Löf randomness for Brownian motion in the usual way: the Martin-Löf random elements of $(C[0,1],\mathbb{P})$ are those which do not belong to the universal Martin-Löf test $\bigcap_n \mathcal{U}_n$. To stress the difference between Brownian motion as a stochastic process and Martin-Löf randomness on the space $(C[0,1],\mathbb{P})$, we will use the cursive letter $\mathscr{B}$ for the random variable taking values in $C[0,1]$ and distributed according to $\mathbb{P}$, and use the letter B for individual elements of $C[0,1]$. Recall that we refer to elements $B\in C[0,1]$ as (sample) paths, and therefore we will only talk about *Martin-Löf random paths*, and not Martin-Löf random Brownian motion.

One can extend Brownian motion to $C[0,\infty)$ in the following way. Let $\{\mathscr{B}_n(t)\}_{n\in\mathbb{N}}$ be independent Brownian motions on $C[0,1]$. Then

$$\mathscr{B}(t)=\mathscr{B}_{\lfloor t\rfloor}(t-\lfloor t\rfloor)+\sum_{0\leq i<\lfloor t\rfloor}\mathscr{B}_i(1) \tag{1}$$

satisfies the definition of Brownian motion for the space of continuous functions $C([0,\infty),\mathbb{R})$. One can adapt the definition of Martin-Löf randomness for Brownian motions over $[0,1]$ to Brownian motion over $[0,\infty)$ in a straightforward way. (For example, by the above correspondence (1) Brownian motion over $[0,\infty)$ can be identified with ω copies of $(C[0,1],\mathbb{P})$, endowed with the product measure $\mathbb{P}^\omega$).

1.3 Layerwise computability

Throughout the paper, we will make extensive use of the notion of *layerwise computability* developed by Hoyrup and Rojas [12, 13]. Layerwise computability is a form of uniform relativisation. In computability theory, we often say that an element y is *computable in* x if y can be computed given x as an oracle. We say that an expression $F(x)$ is *computable uniformly in* x if F is a computable function on the space to which x belongs. There are many examples of this in computable analysis: x^2 is computable uniformly in $x\in[0,1]$, the integral $\int f$ is computable uniformly in $f\in C[0,1]$ (endowed with the $||.||_\infty$ norm), etc.

Layerwise computability is a slightly weaker form of uniformity. First of all when we say that an expression $F(x)$ is layerwise computable, we only ask that it be defined for x Martin-Löf random on the computable probability space $\mathbb{X}$ to which it belongs. (See Hoyrup and Rojas [12, 13] for the definition of computable probability space). Moreover, we only require uniformity on each "layer" of $\mathbb{X}$, uniformly in n. A layer is a set of type $\mathcal{K}_n$, where $\mathcal{K}_n$ is the complement of $\mathcal{U}_n$, the n-th level of a fixed universal Martin-Löf test over $\mathbb{X}$. An interesting aspect of layers is that they always are effectively compact, even if the space $\mathbb{X}$ itself is not compact. So formally, we say that $F(x)$ is computable layerwise in x if there exists a partial computable function $G(.,.)$ such that $G(x,n) = F(x)$ for all $x \in \mathcal{K}_n$.

Layerwise computability is a very powerful tool to study constructive versions of classical results in probability theory and measure theory (as we shall see in this paper!). Perhaps the most important result using layerwise computability is the so-called "randomness preservation theorem":

Theorem 1.1 (Hoyrup and Rojas [12, 13]) *Let $(\mathbb{X}, \mu)$ be a computable probability space and F a layerwise computable function over $\mathbb{X}$ taking values in a computable metric space $\mathbb{Y}$. Then:*

(i) The push forward measure ν defined over $\mathbb{Y}$ by $\nu(\mathcal{A}) = \mu(F^{-1}(\mathcal{A}))$ is computable.

(ii) If x is μ-Martin-Löf random, then $F(x)$ is ν-Martin-Löf random.

(iii) For every $y \in \mathbb{Y}$ which is ν-Martin-Löf random, there is some μ-ML random $x \in \mathbb{X}$ such that $F(x) = y$.

This theorem can for example be used to prove that $C[0,1]$ with the $||.||_\infty$ norm and Wiener measure is a computable probability space (as alluded to in the previous subsection). Indeed, Fouche and Davie proved that the function Φ which maps a sequence of reals $\xi_0, \xi_1, \{\xi_{i,j}\}_{i \in \mathbb{N}, j < 2^i}$ to the function

$$\mathscr{B}(t) = \xi_0 \Delta_0(t) + \xi_1 \Delta_1(t) + \sum_i \sum_{j<2^i} \xi_{i,j} \Delta_{i,j}(t)$$

is layerwise computable from $\mathbb{X}$ to $(C[0,1], ||.||_\infty)$, where $\mathbb{X}$ is the space of sequences of real numbers where each coordinate is distributed according to the normal distribution $\mathcal{N}(0,1)$ independently of the others. It is obvious that $\mathbb{X}$ is a computable probability space. Thus, by the above theorem, the measure induced by Φ on $(C[0,1], ||.||_\infty)$, which we know to be Wiener measure, is a computable measure.

Another important result we will need in several occasions is that one can compute the integral of layerwise computable functions.

Theorem 1.2 (Hoyrup and Rojas [12]) *Let f be a bounded layerwise computable function defined on some computable probability space $(\mathbb{X}, \mu)$. Then the integral*

$$\int_{x \in \mathbb{X}} f(x)\, d\mu(x)$$

is computable uniformly in an index of f and a bound for it.

2 Basic properties of Martin-Löf random paths

We begin by showing that the main "almost sure" properties of classical Brownian motion hold for Martin-Löf random paths.

2.1 Scaling theorem

The classical scaling theorem states that the map $B(t) \mapsto \frac{1}{a}B(a^2t)$ is a Wiener-measure-preserving map from $C[0,1]$ to $C[0,1]$ (or $C[0,\infty) \to C[0,\infty)$), see for example Mörters and Peres [20, Lemma 1.7]. For Martin-Löf random paths, we have the following.

Proposition 2.1 *Let B be a Martin-Löf random path of $C[0,1]$ (resp. of $C[0,\infty)$). Then $\frac{1}{a}B(a^2t)$ is also a Martin-Löf random path of $C[0,1]$ (resp. of $C[0,\infty)$) whenever B is random relative to a.*

Proof The map $B(t) \mapsto \frac{1}{a}B(a^2t)$ is a-computable and measure preserving, therefore it preserves Martin Löf randomness relative to a by Theorem 1.1 relativized to a. $\square$

2.2 Constructive strong Markov property

The strong Markov property of Brownian motion asserts the following. Let T be a stopping time, that is, a random variable in $[0,\infty]$ which is a function of $\mathscr{B}$, and such that deciding whether $\{T \leq t\}$ depends only on $\mathscr{B} \upharpoonright [0,t]$ (the restriction of $\mathscr{B}$ to the interval $[0,t]$). If $T(\mathscr{B})$ is almost surely finite, then the process $\widehat{\mathscr{B}}$ defined by $\widehat{\mathscr{B}}(t) = \mathscr{B}(T(\mathscr{B})+t) - \mathscr{B}(T(\mathscr{B}))$ is a Brownian motion independent of $\mathscr{B} \upharpoonright [0, T(\mathscr{B})]$.

From its classical version, we can derive a constructive version of the strong Markov property which will be very useful in the sequel.

Proposition 2.2 *Let T be an almost surely finite layerwise computable stopping time. Then the function*

$$B \mapsto \widehat{B},$$

where $\widehat{B}(t) = B(T(B)+t) - B(T(B))$, is layerwise computable and if B is Martin-Löf random, then $\widehat{B}$ is Martin-Löf random relative to $B \upharpoonright [0, T(B)]$ and $T(B)$.

Proof Consider the product space $C[0,\infty) \times C[0,\infty)$ endowed with the product measure $W \times W$. Consider the map

$$(B_1, B_2) \mapsto \left((B_1 \upharpoonright [0, T(B_1)])^\frown B_2\,,\, \widehat{B_1}\right)$$

from $C[0,\infty) \times C[0,\infty)$ into itself, where $(B_1 \upharpoonright [0, T(B_1)])^\frown B_2$ is the concatenation of B_1 up to time $T(B_1)$ and then continued according to B_2:

$$\left((B_1 \upharpoonright [0, T(B_1)])^\frown B_2\right)(t) = \begin{cases} B_1(t) \text{ if } t \leq T(B_1) \\ B_1(T(B_1)) + B_2(t - T(B_1)) \text{ if } t \geq T(B_1) \end{cases}$$

By the strong Markov property this map is measure preserving, and it is layerwise computable since $B_1 \mapsto T(B_1)$ is. Thus, if (B_1, B_2) is Martin-Löf random the pair $\left((B_1 \upharpoonright [0, T(B_1)])^\frown B_2\,,\, \widehat{B_1}\right)$ is also Martin-Löf random, and thus by van Lambalgen's theorem $\widehat{B_1}$ is random relative to $(B_1 \upharpoonright [0, T(B_1)])^\frown B_2$. The stopping time of $(B_1 \upharpoonright [0, T(B_1)])^\frown B_2$ is also $T(B_1)$ (by definition of a stopping time), thus $(B_1 \upharpoonright [0, T(B_1)])^\frown B_2$ computes $T(B_1)$ and thus computes $B_1 \upharpoonright [0, T(B_1)]$ (because T is layerwise computable). Therefore, $\widehat{B_1}$ is Martin-Löf random relative to $T(B_1)$ and $B_1 \upharpoonright [0, T(B_1)]$. □

2.3 Continuity properties

In his paper establishing many of the local properties of Martin-Löf random Brownian motion, Fouché [9] shows every Martin-Löf random Brownian motion obeys a modulus of continuity $\phi(h)$ such that

$$\limsup_{h \to 0} \sup_{0 \leq t \leq 1-h} \frac{|B(t+h) - B(t)|}{\phi(h)} \leq 1$$

and

$$\phi(h) = O\left(\sqrt{h \log(1/h)}\right) \tag{2}$$

It is possible to extend this result with big-O notation to the particular constant ($\sqrt{2}$) from the classical result, and moreover, while the classical result demonstrates that the modulus of continuity holds for "sufficiently small" h, we will demonstrate that "sufficiently small" is layerwise computable from a Martin-Löf random path.

Proposition 2.3 *Let B be a Martin-Löf random Brownian motion. Then for all $c < \sqrt{2}$, for all h_0 there exists $h < h_0$ such that*

$$|B(t+h) - B(t)| > c\sqrt{h \log(1/h)}$$

for all t.

Proof For a large n (to be specified later), split the interval $[0, 1]$ into chunks of size e^{-n} (omitting the last bit). For each $0 \leq k < e^n$, consider the event

$$A_k : |B((k+1)e^{-n}) - B(ke^{-n})| \geq c\sqrt{e^{-n}n}$$

(ie, what we want, with $h = e^{-n}$).

Note that the A_k are independent by definition of Brownian motion, and by time-translation invariance all have the same probability. Let us estimate the probability of A_0, which is the event: $|B(e^{-n}) - B(0)| \geq c\sqrt{e^{-n}n}$. By scaling, it is also equal to the probability of the event $|B(1) - B(0)| \geq c\sqrt{n}$. By the estimate given in Mörters and Peres [20, Lemma 12.9], we have

$$\mathbb{P}(A_0) \geq \frac{c\sqrt{n}}{c^2 n + 1} e^{-c^2 n/2}$$

so, by assumption on c, there exists an $\alpha < 1$ such that for almost all n

$$\mathbb{P}(A_0) \geq e^{-\alpha n}$$

Since the A_k are independent,

$$\mathbb{P}(\text{no } A_k \text{ happens}) \leq (1 - e^{-\alpha n})^{e^n} \sim e^{-e^{(1-\alpha)n}}$$

Thus for n taken large enough, this can be made arbitrarily small. Moreover, notice that c can be supposed to be computable, which makes the A_k Π^0_1 classes, hence the event "no A_k happens" corresponds to a Σ^0_1 class. Thus, we have a Solovay test that any Martin-Löf random Brownian motion should pass, and for such a Martin-Löf random Brownian path, there are infinitely many n for which some A_k happens. □

Proposition 2.4 *Let B be a Martin-Löf random Brownian motion. Then for all $c > \sqrt{2}$, there is h_0 such that for all $h < h_0$ and all t,*

$$|B(t+h) - B(t)| \leq c\sqrt{h \log(1/h)}$$

Moreover, such an h_0 can be found uniformly in B and an upper bound on its randomness deficiency.

The proof is the same as that of Mörters and Peres [20, Theorem 1.14] with the addition of keeping track of the layerwise computability of h_0. We recall the proof for completeness.

We first look at increments over a class of intervals which is chosen to be sparse, but big enough to approximate arbitrary intervals. More precisely, given $n, m \in \mathbb{N}$, we let $\Lambda_n(m)$ be the collection of all intervals of the form

$$[(k-1+b)2^{-n+a}, \ \ (k+b)2^{-n+a}],$$

for $k \in \{1, ...2^n\}$, $a, b \in \{0, \frac{1}{m}, ..., \frac{m-1}{m}\}$. We further define $\Lambda(m) \coloneqq \bigcup_n \Lambda_n(m)$.

Lemma 2.5 *For any fixed m and $c > \sqrt{2}$, for B a Martin-Löf random Brownian motion, and an upper bound on the randomness deficiency of B, one can effectively find an $n_0 \in \mathbb{N}$ such that for any $n \geq n_0$,*

$$|B(t) - B(s)| \leq c\sqrt{(t-s)\log\frac{1}{(t-s)}} \qquad \text{for all } [s,t] \in \Lambda_m(n).$$

Proof From the tail estimate for a standard normal variable X, for example Mörters and Peres [20, Lemma 12.9], we obtain

$$\mathbb{P}\Bigg\{ \sup_{k\in\{1,\dots,2^n\}} \sup_{a,b\in\{0,\frac{1}{m},\dots,\frac{m-1}{m}\}} |B((k-1+b)2^{-n+a}) - B((k+b)2^{-n+a})| > c\sqrt{2^{-n+a}\log(2^{n+a})}\Bigg\}$$

$$\leq 2^n m^2 \mathbb{P}\{X > c\sqrt{\log(2^n)}\}$$

$$\leq \frac{m^2}{c\sqrt{\log(2^n)}}\frac{1}{\sqrt{2\pi}}2^{n(1-\frac{c^2}{2})}. \tag{3}$$

Note that c can be taken to be computable, so for fixed $m, n \in \mathbb{N}$ the event

$$\sup_{k\in\{1,\dots,2^n\}} \sup_{a,b\in\{0,\frac{1}{m},\dots,\frac{m-1}{m}\}} |B((k-1+b)2^{-n+a}) - B((k+b)2^{-n+a})| > c\sqrt{2^{-n+a}\log(2^{n+a})}$$

is computable in $B(t)$ and the right hand side of 3 is summable, giving a Solovay test which every Martin-Löf random Brownian motion $B(t)$ will pass.

The standard proof of the equivalence of Solovay randomness and Martin-Löf randomness gives a uniform way of converting a Solovay test $\{\mathcal{S}_i\}$ to a Martin-Löf test $\{\mathcal{U}_j\}$. See, for example, Downey and Hirschfeldt [6]. Thus knowing a k such that a Martin-Löf random path $B(t) \notin \mathcal{U}_k$ gives us an n_0 where the path no longer appears in any $\mathcal{S}_n$ for $n > n_0$. □

Lemma 2.6 *Given* $\varepsilon > 0$ *there exists* $m \in \mathbb{N}$ *such that for every interval* $[s, t] \subset [0, 1]$ *there exists an interval* $[s', t'] \in \Lambda(m)$ *with* $|t - t'| < \varepsilon(t - s)$ *and* $|s - s'| < \varepsilon(t - s)$.

Proof See Mörters and Peres [20, Lemma 1.17]. □

Proof of Proposition 2.4 Given $c > \sqrt{2}$, pick $0 < \varepsilon < 1$ small enough to ensure that $c^* := c - \varepsilon > \sqrt{2}$ and $m \in \mathbb{N}$ as in Lemma 2.6. Using Lemma 2.5 we choose $n_0 \in \mathbb{N}$ large enough that for all $n \geq n_0$ and all intervals $[s', t'] \in \Lambda_n(m)$, almost surely

$$|B(t') - B(s')| \leq c^* \sqrt{(t' - s') \log \frac{1}{(t' - s')}}$$

Now let $[s, t] \subset [0, 1]$ be arbitrary with $t - s < \min(2^{-n_0}, \varepsilon)$, and pick $[s', t'] \in \Lambda(m)$ with $|t - t'| < \varepsilon(t - s)$ and $|s - s'| < \varepsilon(t - s)$. Then, recalling (2), there is a C such that

$$\begin{aligned} |B(t) - B(s)| &\leq |B(t) - B(t')| + |B(t') - B(s')| + |B(s') - B(s)| \\ &\leq C\sqrt{|t - t'| \log \frac{1}{|t - t'|}} + c^* \sqrt{(t' - s') \log \frac{1}{(t' - s')}} + C\sqrt{|s - s'| \log \frac{1}{|s - s'|}} \\ &\leq (4C\sqrt{\varepsilon} + c^* \sqrt{(1 + 2\varepsilon)(1 - \log(1 - 2\varepsilon))}) \sqrt{(t - s) \log \frac{1}{t - s}}. \end{aligned}$$

By making $\varepsilon > 0$ small, the first factor on the right can be chosen arbitrarily close to c. This completes the proof of the theorem. □

2.4 Computability of minimum and maximum

Since a sample path B is almost surely continuous, it almost surely reaches a maximum and a minimum on any given interval. As it turns out, these extremal values can be computed layerwise in B.

Proposition 2.7 *The function*

$$\max(B,x,y) = \max\{B(t) \mid t \in [x,y]\}$$

is computable uniformly in x, y and layerwise in B. The same is true for the minimum function.

Proof To compute the maximum of $B(t)$ on $[x,y]$ to within ε, we run the following simple algorithm: Pick h_0 small enough so that $B(t)$ obeys a modulus of continuity with constant $c = 2$ (see Proposition 2.4) and so that $2\sqrt{h_0 \log(1/h_0)} < \varepsilon$. Then we know that the maximum of the values $B(r_1), B(r_1 + h_0), B(r_1 + 2h_0), \ldots, B(r_2)$ must be within $2\sqrt{h_0 \log(1/h_0)}$, and therefore within ε, of the maximum value of $B(t)$ on $[x,y]$. The minima are also layerwise computable by the same argument. □

Proposition 2.8 *Local maxima and local minima of a Martin-Löf random Brownian motion are Martin-Löf random reals (in particular, they cannot be computable reals).*

Proof Fix two rational numbers $x < y$. It is known classically that $\max(\mathscr{B}, 0, y)$ is distributed according to the density function

$$f(a) = 2 \cdot \frac{e^{-a^2/(2y)}}{\sqrt{2\pi y}}$$

for $a \geq 0$, and $f(a) = 0$ for $a < 0$ (see Mörters and Peres [20, Theorem 2.21]). By the Markov property, $\max(\mathscr{B}, x, y)$ has the same distribution as $\mathscr{B}(x) + \max(\mathscr{B}', 0, y - x)$, where $\mathscr{B}$ and $\mathscr{B}'$ are two independent Brownian motions, and thus is distributed according to the convolution $f * g$ of the functions

$$f(a) = 2\frac{e^{-a^2/(2x)}}{\sqrt{2\pi x}}$$

and

$$g(a) = 2\frac{e^{-a^2/(2(y-x))}}{\sqrt{2\pi(y-x)}}$$

for $a \geq 0$, and $f(a) = 0$ for $a < 0$. This convolution product is computable (it is just an integral of a product of two computable functions), and it is known that if a measure μ on $\mathbb{R}$ admits a computable positive density function, then its random elements are exactly the Martin-Löf random reals for the uniform measure (see Hoyrup and Rojas [13]). □

Corollary 2.9 *If a Martin-Löf random B has a zero on some interval $[a,b]$, there are $x, y \in [a,b]$ such that $f(x) > 0$ and $f(y) < 0$.*

Proof Otherwise 0 would be a local maximum or minimum, which would contradict Proposition 2.8. □

3 Zero sets of Martin-Löf random Brownian motion

In this section, we study the properties of the zero set

$$Z_B = \{t \geq 0 : B(t) = 0\}$$

of Martin-Löf random paths. Once again, we will need some classical results to prove our effective theorems. Most importantly, we will need the next proposition, which gives an exact expression of the probability that a path has a zero in a given interval.

Proposition 3.1 (see Peres [22]) *For any $\varepsilon \in (0,1)$ and $a > 0$*

$$\mathbb{P}_0\Big(B(s) = 0 \text{ for some } s \in [a, a+\varepsilon]\Big) = \frac{2}{\pi} \arctan\left(\sqrt{\frac{\varepsilon}{a}}\right)$$

which is $\sim \frac{2}{\pi}\sqrt{\frac{\varepsilon}{a}}$ as ε tends to 0.

We shall also need the following lemma.

Lemma 3.2 *Let $[a,b]$ be a sub-interval of $[0,\infty)$. Then for all x*

$$\mathbb{P}_0\big(\mathscr{B} \text{ has a zero in } [a,b]\big) \geq \mathbb{P}_x\big(\mathscr{B} \text{ has a zero in } [a,b]\big)$$

Proof Consider the random variable $\mathscr{B}$ consisting of a Brownian motion starting at 0, and form the variable $\mathscr{B}'$ defined as follows: $\mathscr{B}'(t) = x - \mathscr{B}(t)$ for $t \leq \tau$ and $\mathscr{B}'(t) = \mathscr{B}(t)$ for $t \geq \tau$, where τ is the first time s at which $\mathscr{B}(s) = x - \mathscr{B}(s)$. Then the distribution of $\mathscr{B}'$ is that of a Brownian motion starting at x. Moreover, if $\mathscr{B}'(t) = 0$ for some $t \in [a,b]$, then by continuity we have $\tau < t$, and thus $\mathscr{B}(t) = \mathscr{B}'(t) = 0$. This shows that

$$\mathbb{P}(\mathscr{B}' \text{ has a zero in } [a,b]\big) \leq \mathbb{P}\big(\mathscr{B} \text{ has a zero in } [a,b]\big)$$

and the result follows. □

3.1 The zero set of B is layerwise recursive in B

Following Weihrauch [26, Definition 5.1.1], we say that a closed set $\mathcal{C}$ is *recursive* if the predicates

$$\mathcal{C} \cap [a,b] = \emptyset$$

and

$$\mathcal{C} \cap (a,b) \neq \emptyset$$

over a pair (a, b) of rationals, are both Σ^0_1. More generally, if $\mathcal{C}$ is a subset of $\mathbb{R}^d$, we say that it is recursive if one can enumerate the open balls of rational center and rational radius which intersect $\mathcal{C}$ and one can enumerate the closed balls of rational center and rational radius which are disjoint from $\mathcal{C}$.

Remark 3.3 Note that a recursive closed set is in particular a Π^0_1 class. Not all Π^0_1 classes are recursive. For example, the minimum element of a bounded recursive closed set is necessarily a computable real, a property that not all bounded Π^0_1 subsets of $\mathbb{R}$ have. To see this, suppose without loss of generality that all members of $\mathcal{C}$ are positive. Then the minimum is lower semicomputable as

$$\min(\mathcal{C}) = \sup\{q \in \mathbb{Q} \mid (0, q) \cap \mathcal{C} = \emptyset\}$$

and upper semicomputable as

$$\min(\mathcal{C}) = \inf\{q \in \mathbb{Q} \mid \exists q' \in \mathbb{Q}\ (q', q) \cap \mathcal{C} \neq \emptyset\}$$

The main result of this subsection is that the zero set Z_B is recursive layerwise in B. To prove this fact, we first need to show the following proposition.

Proposition 3.4 *For B Martin-Löf random, the origin is not an isolated zero.*

Proof For all k, we know from Proposition 3.1 that the probability for Brownian motion not having a zero on the interval $(2^{-3k}, 2^{-3k} + 2^{-k})$ is

$$1 - \frac{2}{\pi} \arctan(2^k)$$

which limits to zero, computably, as $k \to \infty$. Moreover, we argued above that not having a zero in a given rational interval is a Σ^0_1 event, thus this gives us a Martin-Löf test (in fact, a Schnorr test), and thus a Martin-Löf random B must have a zero in infinitely many intervals of type $(2^{-3k}, 2^{-3k} + 2^{-k})$. □

Proposition 3.5 *For B Martin-Löf random, the set Z_B does not contain any computable real other than 0.*

Proof Suppose $x > 0$ is computable. Let $[a_k, a_k + 2^{-k}]$ be a computable sequence of rational intervals containing x. The probability for $\mathscr{B}$ to have a zero in $[a_k, a_k + 2^{-k}]$ is $O(2^{-k/2})$ (the multiplicative constant depending on x), and by Corollary 2.8 having a zero in $[a_k, a_k + 2^{-k}]$ for a Martin-Löf random Brownian motion is equivalent to having a positive and a negative value on $[a_k, a_k + 2^{-k}]$, which is a Σ^0_1 property. Therefore this induces a Martin-Löf test, and thus any Martin-Löf random B must have no zero in $[a_k, a_k + 2^{-k}]$ for some k. □

Theorem 3.6 *For B a Martin-Löf random path, Z_B is a non-empty closed set which is recursive layerwise in B.*

Proof Z_B is closed because $B(t)$ is continuous. Let us now prove that Z_B is recursive layerwise in B. Since we already know that no rational can be a zero of Bn all we need to show is that one can decide, layerwise in B, whether B has a zero in a rational interval (a,b) with $a < b$. If $a = 0$, we know by Proposition 3.4 that answer is necessarily yes, so we can assume $a > 0$. By Corollary 2.9, in case B does have a zero on (a,b), it must take a positive and a negative value somewhere on the interval. Conversely, having a positive and a negative value on the interval guarantees the existence of zero. Since having a positive and a negative value is a Σ^0_1 event, the predicate $\mathcal{C} \cap (a,b) \neq \emptyset$ is itself Σ^0_1, uniformly in B. It remains to show that $\mathcal{C} \cap (a,b) = \emptyset$ is Σ^0_1 layerwise in B. Note that by Proposition 3.5, B cannot have a zero at a nor at b, so

$$\mathcal{C} \cap (a,b) = \emptyset \Leftrightarrow \max(B,a,b) > 0 \text{ or } \min(B,a,b) < 0$$

Since $\max(B,a,b)$ and $\min(B,a,b)$ are layerwise computable in B, this shows that $\mathcal{C} \cap (a,b) = \emptyset$ is a Σ^0_1 predicate. □

This theorem yields several useful corollaries.

Corollary 3.7 *The first zero of B on an interval $[a,b]$ with $a < b$ rationals (taking value $\perp$ if there is no such zero) is computable layerwise in B and uniformly in a,b.*

Proof Again, note that if $a = 0$, then the first zero is 0. Now, suppose $a > 0$. By Proposition 3.5 B cannot have a zero at a nor at b, thus $Z_B \cap [a,b] = Z_B \cap (a,b)$, and one can immediately check whether the latter is empty (layerwise in B) since Z_B is recursive layerwise in B. In the case $Z_B \cap [a,b] \neq \emptyset$ we have explained in Remark 3.3 that the minimum of a recursive closed set can be computed (uniformly in a code for this closed set). It is easy to see that $Z_B \cap [a,b]$ is itself recursive layerwise in B and uniformly in a,b, thus its minimum element can be computed layerwise in B and uniformly in a,b. □

Corollary 3.8 *If F be is a finite union of rational intervals, $\mathbb{P}\{Z_{\mathscr{B}} \cap F \neq \emptyset\}$ is computable uniformly in a code for F. If U is an effectively open subset of $[0,1]$, then $\mathbb{P}\{Z_{\mathscr{B}} \cap U \neq \emptyset\}$ is lower semi-computable uniformly in an index for U.*

Proof For a given F, let $\mathcal{E}_F$ be the event $[Z_{\mathscr{B}} \cap F \neq \emptyset]$. By Theorem 3.6, the characteristic function $\mathbf{1}_{\mathcal{E}_F}$ is layerwise computable, uniformly in a code for F. Thus, by Theorem 1.2

$$\mathbb{P}[Z_{\mathscr{B}} \cap F \neq \emptyset] = \int_B \mathbf{1}_{\mathcal{E}_F}(B)\, d\mathbb{P}(B)$$

is computable uniformly in a code for F. To get the lower semi-computability of $\mathbb{P}\{Z_{\mathscr{B}} \cap U\} \neq \emptyset\}$ when U is an effectively open set, it suffices to observe that

$$\mathbb{P}[Z_{\mathscr{B}} \cap U \neq \emptyset] = \sup_t \mathbb{P}[Z_{\mathscr{B}} \cap U[t] \neq \emptyset]$$

where $U[t]$ is the approximation of U at stage t, which is a finite union of rational intervals. $\square$

Finally, we show that Z_B has no isolated point for B Martin-Löf random.

Proposition 3.9 *For B Martin-Löf random, Z_B has no isolated point.*

Proof Consider $\tau_q = \inf\{t \geq q : B(t) = 0\}$, the first zero after some $q \in \mathbb{Q}$. By closure of Z_B, the infimum is a minimum. Moreover, τ_q is layerwise computable in B by Corollary 3.7 and is an almost surely finite stopping time. Thus by the constructive strong Markov property, and Proposition 3.4, τ_q is not an isolated zero from the right.

Now, consider zeros that are not of the form τ_q. Call some such zero t_0. To see it is not isolated from the left, consider a sequence of rationals $q_n \uparrow t_0$. By assumption on t_0, for all n there is some $\tau_{q_n} \in (q_n, t_0)$, so t_0 is not an isolated zero from the left. $\square$

3.2 Effective version of Kahane's Theorem

Next, we prove an effective version of the following theorem of Kahane's, which we will need in the next section.

Theorem 3.10 (Kahane [14, p246]) *Let E_1 and E_2 be two (disjoint) closed subsets of $[0, 1]$ such that $\dim(E_1 \times E_2) > 1/2$ then:*

$$\mathbb{P}(B[E_1] \cap B[E_2] \neq \emptyset) > 0$$

(where $B[E]$ is the set $\{B(t) : t \in E\}$ and dim denotes Hausdorff dimension). We shall prove the following.

Theorem 3.11 *Let E_1 and E_2 be two (disjoint) Π^0_1 classes such that $\dim(E_1 \times E_2) > 1/2$; then:*

(i) There exists a Martin-Löf random path B such that $B[E_1] \cap B[E_2] \neq \emptyset$

(ii) Given a fixed Martin-Löf random path B, there exists an integer c such that $B[E_1/c] \cap B[E_2/c] \neq \emptyset$

Proof First of all, observe that item (i) of the theorem follows from item (ii). Indeed, if we have a Martin-Löf random path B and an integer c such that $B[E_1/c] \cap B[E_2/c] \neq \emptyset$, by the scaling property $\frac{1}{\sqrt{c}}B(ct)$ is also Martin-Löf random and satisfies (i). Thus we only need to prove (ii). For this we will use the classical version of theorem (Kahane's) theorem, together with Blumenthal's 0-1 law and some recent results of algorithmic randomness. Recall that Blumenthal's 0-1 law states that any event which only depends on a infinitesimal time interval on the right of the origin (formally, any event in the σ-algebra $\bigcap_{s>0} \sigma\{\mathscr{B}(t) : 0 \leq t \leq s\}$) has probability either zero or one (see Mörters and Peres [20, Theorem 2.7]).

Consider the scaling map $S : B \mapsto \frac{1}{2}B(4t)$. As we saw in Subsection 2.1, S is computable and preserves Wiener measure $\mathbb{P}$ on $C[0,1]$. Moreover, this map is *ergodic*. Indeed, let $\mathcal{A}$ be an $\mathbb{P}$-measurable event which is invariant under S, i.e we have $B \in \mathcal{A} \Leftrightarrow S(B) \in \mathcal{A}$. By induction, $B \in \mathcal{A} \Leftrightarrow \forall n\, S^n(B) \in \mathcal{A}$. The function $S^n(B)$ on $[0,1]$ only depends on the values of B on $[0, 4^{-n}]$. Therefore the event $\mathcal{A}$, which is equal to $[\forall n\, S^n(B) \in \mathcal{A}]$, only depends on the germ of B. By Blumenthal's 0-1 law, this ensures that $\mathcal{A}$ has probability 0 or 1. Thus S is ergodic.

Now, consider the set

$$\mathcal{U} = \{B \mid B[E_1] \cap B[E_2] = \emptyset\}$$

We claim that $\mathcal{U}$ is a Σ^0_1 subset of $C([0,1])$. This is because of a classical result in computable analysis: the image of a bounded Π^0_1 class by a computable function is a bounded Π^0_1 class. This fact is uniform: from an index of a Π^0_1 class P and a computable function f one can effectively compute the index of the Π^0_1 class $f[P]$. By uniform relativization, there is a computable function γ s.t. given a pair (f, P) where f is a continuous function given as oracle, and P is a Π^0_1 class of index e, $\gamma(e)$ is an index for $f[P]$ as a $\Pi^{0,f}_1$-class. Moreover, one can effectively compute an upper and lower bound a, b for $f[P]$ from f and P. Here we have two Π^0_1 classes E_1 and E_2, say of respective indices e_1 and e_2. By the above discussion $B[E_1]$ and $B[E_2]$ have respective indices $\gamma(e_1)$ and $\gamma(e_2)$ as $\Pi^{0,B}_1$-classes, and since the intersection of two Π^0_1 classes is index-computable there is a computable function θ such that $B[E_1] \cap B[E_2]$ has index $\theta(e_1, e_2)$ as a $\Pi^{0,B}_1$-class. Since one can computably enumerate, uniformly in the oracle B, the indices of $\Pi_1^{0,B}$-classes whose intersection with $[a, b]$ is empty, it follows that the set $\mathcal{U}$ is Σ_1^0, as wanted.

We can now apply the effective ergodic theorem proven in Bienvenu et al. [3] and independently Franklin et al. [10]: since $\mathcal{U}$ has measure less than 1 (by Kahane's theorem) and is a Σ_1^0 set, there are infinitely many n such that $S^n(B) \notin \mathcal{U}$ (in fact, the set of such n's is a subset of $\mathbb{N}$ of positive density), ie such that $B[E_1/2^n] \cap B[E_2/2^n] \neq \emptyset$. □

4 The effective dimension of zeros

Effective Hausdorff dimension is a modification of Hausdorff dimension for the computability setting. Intuitively, effective Hausdorff dimension describes how "computably locatable" a point or set is in addition to its size. For example, an algorithmically random point in $\mathbb{R}^n$ has effective Hausdorff dimension n because it can't be computably located any more precisely than a small computable ball, which has Hausdorff dimension n.

There are many equivalent definitions of effective Hausdorff dimension, but we will use the following definition of Mayordomo[19]. See the book by Downey and Hirschfeldt [6], or papers by Lutz [18] and Reimann [23, 25] for more details.

Definition 4.1 The *effective (or constructive) Hausdorff dimension* of $X \in 2^\omega$ is

$$\operatorname{cdim}(x) := \liminf_n \frac{K(X \upharpoonright n)}{n}$$

This definition can be extended to real numbers by identifying them with their binary representation.

In this section, we will try to characterize the effective dimension of the zeroes of Martin-Löf random paths. This can be broken down in two questions:

(1) Given a Martin-Löf random B, what is the set $\{\operatorname{cdim}(x) \mid x > 0 \text{ and } x \in Z_B\}$?

(2) Given a real x, can we give a necessary or sufficient condition in terms of the effective dimension of x for the existence of some Martin-Löf random path which has a zero at x?

As to the first question, Kjos-Hanssen and Nerode [16] have showed that with probability 1 over B, $\{\mathrm{cdim}(x) \mid x > 0 \text{ and } x \in Z_B\}$ is dense in $[1/2, 1]$[1]. We make this more precise by showing that for every Martin-Löf random path B (not just almost all paths) $\{\mathrm{cdim}(x) \mid x > 0 \text{ and } x \in Z_B\}$ is contained in $[1/2, 1]$ and contains all the computable reals $> 1/2$ of this interval.

We will answer the second question by proving that having effective dimension at least $1/2$ is necessary, while having effective dimension strictly greater than $1/2$ is sufficient (but having dimension exactly $1/2$ is not sufficient).

4.1 The dimension spectrum of Z_B

The next theorem is a direct consequence of the effective version of Kahane's theorem.

Theorem 4.2 *Given a Martin-Löf random path B and computable real $\alpha > 1/2$, there exists a real x in Z_B of constructive dimension α.*

Proof Let B be such a path and α such a real. Consider the Bernoulli measure μ_p (ie measure where each bit has probability p of being a zero, independently of all other bits) such that $p < 1/2$ and $-p \log p - (1-p)\log(1-p) = \alpha$. Since α is computable, so is p (and hence μ_p), because the function $x \mapsto -x \log x - (1-x)\log(1-x)$ is computable and increasing on $[0, 1/2]$. Let $E_1 = \{0\}$ and E_2 be the complement of the first level of the universal Martin-Löf test for μ_p (it is a Π^0_1 class since μ_p is computable). It is well-known that every set of positive μ_p-measure has Hausdorff dimension $\geq \alpha$, and moreover that every μ_p random real has constructive Hausdorff dimension α (see for example Reimann [23]). Applying Theorem 3.11, there exists some c such that $B[E_1/2^c] \cap B[E_2/2^c] \neq \emptyset$. That is, there is some $x \in E_2$ such that $B(2^c x) = 0$. Multiplying by 2^c just adds c zeros in the binary expansion of x, thus $2^c x$ has the same constructive dimension as x, which is α. □

Question 1 The previous theorem could be strengthened with some additional effort to $\mathbf{0}'$-computable α. However, we conjecture that a stronger result is true, namely that for every Martin-Löf random B, it holds that

$$\{\mathrm{cdim}(x) \mid x > 0 \text{ and } x \in Z_B\} = [1/2, 1]$$

We do not know how to show this and leave it as an open question.

[1] This is actually a stronger form of the theorem proven in Kjos-Hnassen and Nerode [16], but the proof of the latter can easily be adapted

4.2 Being a zero of an Martin-Löf random path

We now address the second of the two above questions: what properties (in terms of effective dimension or Kolmogorov complexity) characterize the reals that belong to Z_B for some Martin-Löf random B? To do so, we largely borrow from the work of Kjos-Hanssen [15], but with a number of necessary adaptations to Brownian motion. (Kjos-Hanssen [15] studies a different stochastic process, namely random closed sets, a particular type of percolation limit sets.) Proposition 3.1 gives us a precise expression for the probability of a Brownian motion $\mathscr{B}$ to have a zero in a given interval. The key step needed to adapt Kjos-Hanssen's techniques is to estimate the probability for $\mathscr{B}$ to have a zero in each of *two* intervals of the same length.

Proposition 4.3 *Let $0 < a < b < 1$ and $\varepsilon > 0$. Suppose that the intervals $[a, a+\varepsilon]$ and $[b, b+\varepsilon]$ are disjoint. Let δ be the distance between them (ie, $\delta = b - a - \varepsilon$). Let $\mathcal{A}_1$ be the event "$\mathscr{B}(s) = 0$ for some $s_1 \in [a, a+\varepsilon]$" and $\mathcal{A}_2$ be "$\mathscr{B}(s) = 0$ for some $s_2 \in [b, b+\varepsilon]$". Then*

$$\mathbb{P}_0\left(\mathcal{A}_1 \wedge \mathcal{A}_2\right) \leq \frac{\varepsilon \cdot O(1)}{\sqrt{a\delta}}$$

where the term $O(1)$ is a constant independent of a, b, ε.

Proof In this proof, we make use of the following notation: given an event $\mathcal{A}$, $\mathcal{A}^{\uparrow\tau}$ is the unique (by assumption on $\mathcal{A}$) event such that $t \mapsto B(t+\tau) \in \mathcal{A}^{\uparrow\tau}$ if and only if $t \mapsto B(t) \in \mathcal{A}$.

Now, let $\mathcal{A}_1$ and $\mathcal{A}_2$ be the above events, and let us write

$$\mathbb{P}_0\left(\mathcal{A}_1 \wedge \mathcal{A}_2\right) = \mathbb{P}_0(\mathcal{A}_1)\mathbb{P}_0(\mathcal{A}_2 \mid \mathcal{A}_1)$$

The term $\mathbb{P}_0(\mathcal{A}_1)$ is, by Proposition 3.1, equal to $O(\sqrt{\frac{\varepsilon}{a}})$. It remains to evaluate the term $\mathbb{P}(\mathcal{A}_2 \mid \mathcal{A}_1)$. The event $\mathcal{A}_2$ only depends on the values of $\mathscr{B}$ on the interval $[b, b+\varepsilon]$, thus

$$\mathbb{P}_0(\mathcal{A}_2 \mid \mathcal{A}_1) = \int_{z\in\mathbb{R}} \mathbb{P}_z(\mathcal{A}_2^{\uparrow(a+\varepsilon)}) f(z)\, dz$$

where f is the density function of $\mathscr{B}(a+\varepsilon)$ conditioned by $\mathcal{A}_1$. By shift invariance of the Wiener measure, we observe that in this expression, the term $\mathbb{P}_z(\mathcal{A}_2^{\uparrow(a+\varepsilon)})$ is equal to $\mathbb{P}_z(\mathscr{B}$ has a zero in $[\delta, \delta+\varepsilon])$. This is, in turn, always bounded by

$\mathbb{P}_0(\mathscr{B}$ has a zero in $[\delta, \delta+\varepsilon])$, by Lemma 3.2. Thus

$$\begin{aligned}
\mathbb{P}_0(\mathcal{A}_2 \mid \mathcal{A}_1) &= \int_{z\in\mathbb{R}} \mathbb{P}_z(\mathcal{A}_2^{\uparrow(a+\varepsilon)}) f(z)\, dz \\
&\leq \int_{z\in\mathbb{R}} \mathbb{P}_0(\mathcal{A}_2^{\uparrow(a+\varepsilon)}) f(z)\, dz \\
&\leq \mathbb{P}_0(\mathcal{A}_2^{\uparrow(a+\varepsilon)}) \\
&\leq \mathbb{P}_0(\mathscr{B} \text{ has a zero in } [\delta, \delta+\varepsilon]) \\
&\leq \frac{2}{\pi} \arctan\left(\sqrt{\frac{\varepsilon}{\delta}}\right) \\
&\leq \frac{2}{\pi}\sqrt{\frac{\varepsilon}{\delta}}
\end{aligned}$$

We have thus established the desired result. □

4.2.1 A necessary and a sufficient condition

Our next theorem gives a necessary condition for a point to be a zero of some Martin-Löf random path.

Theorem 4.4 *If B is a Martin-Löf random path, then all members of the set $Z_B \setminus \{0\}$ have effective dimension at least $1/2$.*

Proof Suppose that for a given B, we have $B(a) = 0$ for some a such that $\mathrm{cdim}(a) < 1/2$. We will show that B is not Martin-Löf random.

Let $\mathrm{cdim}(a) < \rho < 1/2$. Take also some rational b such that $0 < b < a$. By definition of constructive dimension, for all n, there exists a prefix σ of a such that $K(\sigma) \leq \rho|\sigma| - n$. For all strings σ such that $0.\sigma > b$, let $I_\sigma = [0.\sigma, 0.\sigma + 2^{-|\sigma|}]$ and the event

$$\mathcal{E}_\sigma : \left[\mathscr{B} \text{ has a positive and a negative value in } I_\sigma\right]$$

The event $\mathcal{E}_\sigma$ is a Σ^0_1 subset of $C[0,1]$, uniformly in σ the probability of $\mathcal{E}_\sigma$ is $O(2^{-|\sigma|/2})$ by Proposition 3.1 (the multiplicative constant depending on b). Define

$$\mathcal{U}_n = \bigcup \{\mathcal{E}_\sigma \mid K(\sigma) \leq \rho|\sigma| - n\}$$

By assumption, B belongs to almost all $\mathcal{U}_n$. However, we have

$$\begin{aligned}\mathbb{P}(B \in \mathcal{U}_n) &\leq O(1) \cdot \sum \{2^{-|\sigma|/2} \mid K(\sigma) \leq \rho|\sigma| - n\} \\ &\leq O(1) \cdot \sum_{\sigma} 2^{-K(\sigma)-n} \\ &\leq O(2^{-n})\end{aligned}$$

Thus the $\mathcal{U}_n$ form a Martin-Löf test, which shows that B is not Martin-Löf random. □

We now prove an (almost) counterpart of Theorem 4.4:

Theorem 4.5 *Let $x \in [0,1]$ be of effective dimension strictly greater than $1/2$. Then there exists a Martin-Löf random path B such that $B(x) = 0$.*

The proof is much more difficult and involves the notion of α-energy. Given a measure μ on $\mathbb{R}$ and $\alpha \geq 0$, the α*-energy* of μ is the quantity

$$\int \int \frac{d\mu(x)d\mu(y)}{|x-y|^{\alpha}}$$

This quantity might be finite or infinite, depending on the value of α. We will need the following two lemmas.

Lemma 4.6 *Let $\beta > \alpha \geq 0$. If μ is a measure satisfying $\mu(A) \leq c \cdot |A|^{\beta}$ for every interval A (or equivalently, for every dyadic interval) and for some constant c, then μ has finite α-energy.*

Proof See Mörters and Peres [20, proof of Theorem 4.32]. □

Lemma 4.7 *2 Let $\beta \geq 1/2$ and let μ be a finite Borel measure on $[0,1]$ such that for every dyadic interval I, $\mu(I) \leq c \cdot |I|^{\beta}$ for some fixed constant c. Suppose that μ has finite $1/2$-energy (which is automatically verified when $\beta > 1/2$). Then there exists a constant $c' > 0$ such that the following holds: for any set $A \subseteq [1/2, 1]$ which is a countable union of closed dyadic intervals*

$$\mathbb{P}_0\big(Z_{\mathscr{B}} \cap A \neq \emptyset\big) \geq c' \cdot \mu(A)^2$$

Proof It suffices to prove this theorem for a finite number of intervals, and up to splitting them if necessary we can assume that they all have the same length 2^{-n} for some n. Let $I_1, ..., I_k$ be those intervals. Define for all k the random variable X_k by

$$X_k = \mu(I_k) \cdot 2^{(n/2)} \cdot \mathbf{1}_{\{Z_{\mathscr{B}} \cap I_k \neq \emptyset\}}$$

and $Y = \sum_{j=1}^{k} X_j$. We want to show that $\mathbb{P}(Y > 0) \geq \frac{\mu(A)^2}{c_0}$ for constant c_0 which does not depend on A, which immediately gives the result (since $Y > 0$ implies $Z_B \cap A \neq \emptyset$). To do so, we will use the Paley-Zigmund inequality

$$\mathbb{P}(Y > 0) \geq \frac{\mathbb{E}(Y)^2}{\mathbb{E}(Y^2)}$$

Let us evaluate separately $\mathbb{E}(Y)$ and $\mathbb{E}(Y^2)$. We have

$$\begin{aligned}
\mathbb{E}(Y) &= \sum_{j=1}^{k} \mathbb{E}(X_j) \\
&\geq \sum_{j=1}^{k} 2^{(n/2)} \cdot \mu(I_j) \cdot c_1 \cdot (\sqrt{2^{-n}}) \\
&\geq c_1 \sum_{j=1}^{k} \cdot \mu(I_j) \\
&\geq c_1 \cdot \mu(A)
\end{aligned}$$

for some constant $c_1 \neq 0$, the second inequality coming from Proposition 3.1.

Let us now turn to $\mathbb{E}(Y^2)$, which we need to bound by a constant. We have

$$\mathbb{E}(Y^2) = \sum_{\substack{1 \leq i \leq k \\ 1 \leq j \leq k}} \mathbb{E}(X_i X_j)$$

To evaluate this sum, we decompose it into three parts:

$$\mathbb{E}(Y^2) = \sum_{i=1}^{k} \mathbb{E}(X_i^2) \; + \; 2 \sum_{\substack{1 \leq i < j \leq k \\ I_i, I_j \text{ adjacent}}} \mathbb{E}(X_i X_j) \; + \; 2 \sum_{\substack{1 \leq i < j \leq k \\ I_i, I_j \text{ nonadjacent}}} \mathbb{E}(X_i X_j)$$

The first part is an easy computation. For all i,

$$\begin{aligned}
\mathbb{E}(X_i^2) &= \mu(I_i)^2 \cdot 2^n \cdot \mathbb{P}\{Z_{\mathscr{B}} \cap I_i \neq \emptyset\} \\
&= O\big(\mu(I_i)^2 \cdot 2^n \cdot 2^{-(n/2)}\big) \\
&= O\big(\mu(I_i) \cdot 2^{-\beta n} \cdot 2^n \cdot 2^{-(n/2)}\big) \\
&= \mu(I_i) \cdot O\big(2^{(1/2-\beta)n}\big) \\
&= \mu(I_i) \cdot O(1)
\end{aligned}$$

(for the third equality, we use the fact that $\mu(I_i) \leq |I_i|^\beta$, and for the fifth one the fact that $\beta \geq 1/2$). Thus

$$\sum_{i=1}^{k} \mathbb{E}(X_i^2) = \sum_{i=1}^{k} \mu(I_i) \cdot O(1) = O(1)$$

For the second part, we use a rough estimate: first notice that

$$\mathbb{E}(X_i X_j) = \mu(I_i) \cdot \mu(I_j) \cdot 2^n \cdot \mathbb{P}\{Z_{\mathscr{B}} \cap I_i \neq \emptyset \ \wedge \ Z_{\mathscr{B}} \cap I_j \neq \emptyset\}$$

and for the second part only, we will use the trivial upper bound:

$$\mathbb{P}\{Z_{\mathscr{B}} \cap I_i \neq \emptyset \ \wedge \ Z_{\mathscr{B}} \cap I_j \neq \emptyset\} \leq \mathbb{P}\{Z_{\mathscr{B}} \cap I_i \neq \emptyset\} = O(2^{-n/2})$$

Combining this with $\mu(I_j) \leq 2^{-\beta n}$, we get:

$$\mathbb{E}(X_i X_j) = \mu(I_i) \cdot O(2^{(1/2-\beta)n}) = \mu(I_i) \cdot O(1)$$

Moreover, each interval I_i has at most two adjacent intervals I_j. Thus,

$$\sum_{\substack{1 \leq i < j \leq k \\ I_i, I_j \text{ adjacent}}} \mathbb{E}(X_i X_j) \leq 2 \sum_{i=1}^{k} \mu(I_i) \cdot O(1) = O(1)$$

Finally, for the third part, we will use the fact that the $1/2$-energy of μ is finite. Let us, for a pair of nonadjacent intervals I_i, I_j with $\max(I_i) < \min(I_j)$, denote by $g(i,j)$ the length of the gap between the two, ie $g(i,j) = \min(I_j) - \max(I_i)$. We have

(4)
$$\sum_{\substack{1 \leq i < j \leq k \\ I_i, I_j \text{ nonadjacent}}} \mathbb{E}(X_i X_j) = \sum_{\substack{1 \leq i < j \leq k \\ I_i, I_j \text{ nonadjacent}}} \mu(I_i) \cdot \mu(I_j) \cdot 2^n \cdot \mathbb{P}\{Z_{\mathscr{B}} \cap I_i \neq \emptyset \ \wedge \ Z_{\mathscr{B}} \cap I_j \neq \emptyset\}$$

By Proposition 4.3,

$$\mathbb{P}\{Z_{\mathscr{B}} \cap I_i \neq \emptyset \ \wedge \ Z_{\mathscr{B}} \cap I_j \neq \emptyset\} = \frac{2^{-n} \cdot O(1)}{\sqrt{g(i,j)}} \tag{5}$$

(Note that we use the fact that I_i and I_j are contained in $[1/2, 1]$, hence $\min(I_i)$ is bounded away from 0.)

Thus,

$$\sum_{\substack{1 \leq i < j \leq k \\ I_i, I_j \text{ nonadjacent}}} \mathbb{E}(X_i X_j) = \sum_{\substack{1 \leq i < j \leq k \\ I_i, I_j \text{ nonadjacent}}} \frac{\mu(I_i) \cdot \mu(I_j)}{\sqrt{g(i,j)}} \cdot O(1) \tag{6}$$

Note that, since I_i and I_j are non-adjacent dyadic intervals of length 2^{-n}, we have $g(i,j) \geq 2^{-n}$. Therefore, for two reals x, y, if $x \in I_i$ and $y \in I_j$, then $|y-x| \leq 3g(i,j)$. By this observation, we have

$$\sum_{\substack{1 \leq i < j \leq k \\ I_i, I_j \text{ nonadjacent}}} \frac{\mu(I_i) \cdot \mu(I_j)}{\sqrt{g(i,j)}} \leq O(1) \cdot \int \int \frac{d\mu(x) d\mu(y)}{|x-y|^{1/2}} \leq O(1)$$

(the last inequality comes from the hypothesis that the $1/2$-energy of μ is finite).

We have thus established that $\mathbb{E}(Y^2) = O(1)$, which completes the proof. □

Let KM denote the a priori Kolmogorov complexity function (see Downey and Hirschfeldt [6, Section 6.3.2]). Recall that $KM(\sigma) = K(\sigma) + O(\log|\sigma|)$, thus in particular K can be replaced by KM in the definition of effective dimension. The reason we need KM instead of K is the following result of Reimann [24, Theorem 14], which we will apply in the proof of Theorem 4.5: Let z be a real such that $KM(z \upharpoonright n) \geq \beta n - O(1)$. Then, there exists a measure μ such that $\mu(A) = O(|A|^\beta)$ for all intervals A, and such that z is Martin-Löf random for the measure μ.

Proof of Theorem 4.5 Let z be of dimension $\alpha > 1/2$. Let β be a rational such that $1/2 < \beta < \alpha$. Then for almost all n, $KM(z \upharpoonright n) \geq \beta n$. By Reimann's theorem, let μ be a measure such that $\mu(A) = O(|A|^\beta)$ for all intervals A, and such that z is Martin-Löf random for the measure μ.

For all n, let $\mathcal{K}_n$ be the complement of the n-th level of the universal Martin-Löf test over $(C[0,1], \mathbb{P})$ and consider the set

$$\mathcal{U}_n = \{x \mid \forall B \in \mathcal{K}_n \; B(x) \neq 0\}$$

We claim that $\mathcal{U}_n$ is Σ^0_1 uniformly in n, and $\mu(\mathcal{U}_n) = O(2^{-n/2})$. To see that it is Σ^0_1 suppose that $x \in \mathcal{U}_n$, ie $B(x) \neq 0$ for all $B \in \mathcal{K}_n$. The set $\mathcal{K}_n$ being compact (see Section 1), the value of $|B(x)|$ for $B \in \mathcal{K}_n$ reaches a positive minimum. Thus there is a rational a such that $B(x) > a$ for all $B \in \mathcal{K}_n$. By uniform continuity of the members of $\mathcal{K}_n$ (ensured by Proposition 2.3), there is a rational closed interval I containing x such that $|B(t)| > a/2$ for all $t \in I$ and $B \in \mathcal{K}_n$. Thus $\mathcal{U}_n$ is the union of intervals (s_1, s_2) such that $\min\{B(t) : t \in [s_1, s_2]\} > b$ for some rational b and all $B \in \mathcal{K}_n$. Moreover, the condition "$\min\{B(t) : t \in [s_1, s_2]\} > b$ for all $B \in \mathcal{K}_n$" is Σ^0_1, because the function $B \mapsto \min\{B(t) : t \in [s_1, s_2]\}$ is layerwise computable (thus uniformly computable on $\mathcal{K}_n$), and the minimum of a computable function on an effectively compact set is lower semi-computable uniformly in a code for that set. This shows that $\mathcal{U}_n$ is Σ^0_1.

To evaluate $\mu(\mathcal{U}_n)$, let us first observe that by definition of $\mathcal{U}_n$,

$$\mathbb{P}_0(Z_{\mathscr{B}} \cap \mathcal{U}_n \neq \emptyset) \leq \mathbb{P}_0(\mathscr{B} \in \mathcal{K}_n \text{ and } Z_{\mathscr{B}} \cap \mathcal{U}_n \neq \emptyset) + 2^{-n} \leq 2^{-n}$$

Applying Lemma 4.7, it follows that $\mu(\mathcal{U}_n) = O(2^{-n/2})$, as wanted. Since z is Martin-Löf random with respect to μ, it cannot be in all sets $\mathcal{U}_n$, and thus it must be the zero of some Martin-Löf random path. □

4.2.2 The case of points of effective dimension $1/2$

In the previous section we showed that no point of effective dimension less than $1/2$ can be the zero of a Martin-Löf random path, and that every point of dimension greater than $1/2$ is necessarily a zero of some Martin-Löf random path. This leaves open the question of what happens at effective dimension exactly $1/2$. While we do not provide a full answer, we show that among points of effective dimension $1/2$, some are zeros of some Martin-Löf random path, and some are not.

The next theorem, which strengthens Theorem 4.4, gives a necessary condition for a point to be a zero of some Martin-Löf random path.

Theorem 4.8 *If $x > 0$ is a zero of some Martin-Löf random path, then*

$$\sum_n 2^{-K(x\restriction n)+n/2} < \infty$$

It is interesting to notice the parallel with the so-called 'ample excess lemma' (see Downey and Hirschfeldt [6, Theorem 6.6.1]): a real x is Martin-Löf random if and only if $\sum_n 2^{-K(x\restriction n)+n} < \infty$.

Proof The proof is an adaptation of that of Theorem 4.4. First take a rational a such that $0 < a$. We shall prove the lemma for all $x > a$, which will be enough since a is arbitrary. For each string σ consider, like in Theorem 4.4, the interval $I_\sigma = [0.\sigma, 0.\sigma + 2^{-|\sigma|}]$ and the event

$$\mathcal{E}_\sigma : \left[\mathscr{B} \text{ has a positive and a negative value in } I_\sigma\right]$$

Now, consider the function $\mathbf{t}$ defined on $C[0,1]$ by

$$\mathbf{t}(B) = \sum_{\sigma \text{ s.t. } a<0.\sigma} 2^{-K(\sigma)+|\sigma|/2} \cdot \mathbf{1}_{\mathcal{E}_\sigma}(B)$$

The event $\mathcal{E}_\sigma$ is a Σ^0_1 subset of $C[0,1]$, uniformly in σ. Thus the function $\mathbf{t}$ is lower semi-computable. Moreover, the probability of $\mathcal{E}_\sigma$ is $O(2^{-|\sigma|/2})$ by Proposition 3.1 (the multiplicative constant depending on a). Thus the integral of $\mathbf{t}$ is bounded, and therefore $\mathbf{t}$ is an integrable test (see Gács [11]). Let now B be a Martin-Löf random path and suppose $B(x) = 0$ for some $x > a$. Then for almost all n, $a < 0.(x \restriction n)$. Moreover, for every n, B having a zero in $I_{x\restriction n}$, it must in fact have a positive and a negative value on that interval (by Proposition 2.8). Thus, by definition of $\mathbf{t}$

$$\mathbf{t}(B) + O(1) \geq \sum_n 2^{-K(x\restriction n)+n/2}$$

(the $O(1)$ accounts for the finitely many terms such that $a \geq 0.(x \restriction n)$). But since B is Martin-Löf random and $\mathbf{t}$ is a integrable test, we have $\mathbf{t}(B) < \infty$, which proves our result. $\square$

This theorem shows in particular that if x is the zero of some Martin-Löf random path, then $K(x \restriction n) - n/2 \to +\infty$.

We now give a sufficient condition which actually is very close to our necessary condition.

Theorem 4.9 *Let $f : \mathbb{N} \to \mathbb{N}$ be a nondecreasing computable function such that $f(n+1) \leq f(n) + 1$ for all n, and such that $\sum_n 2^{-f(n)} < \infty$. Let x be a real such that $KM(x \restriction n) \geq n/2 + f(n) + O(1)$. Then x is the zero of some Martin-Löf random path.*

For this, we need the following refinement of Lemma 4.6.

Proposition 4.10 *Let $f : \mathbb{N} \to \mathbb{N}$ be a function such that $\sum_n 2^{-f(n)} < \infty$. Let μ be a Borel measure on $[0,1]$ such that for every interval A of length $\leq 2^{-n}$, $\mu(A) \leq 2^{-\alpha n - f(n)}$. Then μ has finite α-energy.*

Proof For now, let us fix some x. Define for all n the interval I_n to be $[x - 2^{-n+1}, x - 2^{-n}] \cap [0,1]$ and $J_n = [x + 2^{-n}, x + 2^{-n+1}] \cap [0,1]$. Then

$$\begin{aligned}
\int \frac{d\mu(y)}{|x-y|^\alpha} &\leq \sum_n \int_{y\in I_n} \frac{d\mu(y)}{|x-y|^\alpha} + \sum_n \int_{y\in J_n} \frac{d\mu(y)}{|x-y|^\alpha} \\
&\leq \sum_n 2^{\alpha n}\mu(I_n) + \sum_n 2^{\alpha n}\mu(J_n) \\
&\leq \sum_n 2^{\alpha n} 2^{-\alpha n - f(n)} + \sum_n 2^{\alpha n} 2^{-\alpha n - f(n)} \\
&\leq 2 \cdot \sum_n 2^{-f(n)} \\
&< \infty
\end{aligned}$$

Therefore, the μ-integral over x of $\int \frac{d\mu(y)}{|x-y|^\alpha}$ is itself finite, which is what we wanted. $\square$

Proof of Theorem 4.9 Let f be such a function and x such a real. By a result of Reimann [24, Theorem 14], there exists a measure μ such that $\mu(A) \leq 2^{-n/2-f(n)+O(1)}$ for all intervals of length $\leq 2^{-n}$ such that x is Martin-Löf random with respect to μ. By Proposition 4.10, μ has finite $1/2$-energy. The rest of the proof is identical to the proof of Theorem 4.5. $\square$

Theorem 4.11 *Let $0 < \alpha < 1$ and let $f : \mathbb{N} \to \mathbb{N}$ be a function such that $\sup_n f(n+m) - f(n) = o(m)$. Then there exists $x \in [0,1]$ such that $K(x \upharpoonright n) = \alpha n + f(n) + O(1)$.*

Remark 4.12 This theorem was proven by J. Miller (unpublished) in the particular case where $f = 0$.

Proof Fix a "large enough" integer m, which we will implicitly define during the construction. We will build the sequence x by blocks of length m. For m large enough, the empty string has complexity less than $3\log m$. Suppose we have already constructed a prefix σ of x such that $|K(\sigma \upharpoonright n) - \alpha n - f(n)| \leq 3\log m$ for all $n \leq |\sigma|$ multiple of m. Pick a string τ of length m such that

$$K(\tau \mid \sigma, K(\sigma)) \geq m$$

We then have

$$K(\sigma\tau) \geq K(\sigma) + m - 2\log m - O(1)$$

On the other hand

$$K(\sigma 0^m) \leq K(\sigma) + 2\log m + O(1)$$

For each $i \leq m$, consider the "mixture" between 0^m and τ: $\rho_i = (\tau \upharpoonright i)0^{m-i}$. Since ρ_i and ρ_{i+1} differ by only one bit in position $\leq m$ from the right, we have $|K(\sigma\rho_i) - K(\sigma\rho_{i+1})| \leq 2\log m + O(1)$. By this "continuity" property, setting $N = |\sigma|$, we see that the function $i \mapsto |K(\sigma\rho_i) - \alpha(N+m) - f(N+m)|$ (whose value at $i = 0$ is $-\alpha m + o(m)$ and value at $i = m$ is $(1-\alpha)m - o(m)$), must take a value smaller than $3\log m$, as long as m is chosen large enough. Thus, if m is large enough, we can iterate this argument to build a sequence x such that $|K(x \upharpoonright n) - \alpha n - f(n)| \leq 3\log m$ for all n multiple of m. Since $\alpha n + f(n)$ is a Lipschitz function, this is sufficient to ensure $|K(x \upharpoonright n) - \alpha n - f(n)| = O(1)$. $\square$

We can finally prove the promised theorem.

Theorem 4.13 *Among reals of effective dimension $1/2$, some are zeros of some Martin-Löf random path and some are not.*

Proof By Theorem 4.11, first consider a real x such that $K(x \upharpoonright n) = n/2 + O(1)$. This real has effective dimension $1/2$ and cannot be a zero of a Martin-Löf random path (Theorem 4.8).

Applying Theorem 4.11 again, let y be a real such that $K(y \upharpoonright n) = n/2 + 4\log n + O(1)$. Since for every σ, $KM(\sigma) \geq K(\sigma) - K(|\sigma|) - O(1) \geq K(\sigma) - 2\log|\sigma| - O(1)$, it follows that $KM(y \upharpoonright n) \geq n/2 + 2\log n - O(1)$, and thus y is a zero of some Martin-Löf random path (Theorem 4.9). Of course, y has effective dimension $1/2$ as well. □

This section leaves open the existence of a precise characterization of the reals x of dimension $1/2$ for which there exists a Martin-Löf random path B such that $B(x) = 0$. Short of an exact characterization, it would be interesting to know whether this depends on Kolmogorov complexity alone. By this, we mean the following question.

Question 2 If $K(x \upharpoonright n) \leq K(y \upharpoonright n) + O(1)$ and x is a zero of some Martin-Löf random path, must y be a zero of some Martin-Löf random path? Same question with KM instead of K.

5 Planar Brownian Motion

5.1 Brownian motion in higher dimensions

So far we have talked about Brownian motion on $C[0,1]$ or $C[0,\infty)$, but it is also possible to define Brownian motion in higher dimensions.

Definition 5.1 If $\mathscr{B}_1, \ldots, \mathscr{B}_d$ are independent linear Brownian motions started in $x_1, \ldots, x_d$, then the process $\{\mathscr{B}(t) : t \geq 0\}$ given by $\mathscr{B}(t) = (\mathscr{B}_1(t), \ldots, \mathscr{B}_d(t))$ is *d-dimensional Brownian motion* started in $(x_1, \ldots, x_d)$. The d-dimensional Brownian motion started at the origin is also called *standard Brownian motion*. One-dimensional Brownian motion is also called *linear*, and two-dimensional Brownian motion is also called *planar Brownian motion*.

And similarly, we have

Theorem 5.2 *A function $B(t) = (B_1(t), \dots, B_d(t))$ in the space of continuous functions from $[0, \infty)$ to $\mathbb{R}^d$ with Wiener measure is a Martin-Löf random path if and only if $B_1(t), \dots, B_d(t)$ are mutually Martin-Löf random linear Brownian motion.*

Proof This follows immediately from Van Lambalgen's theorem which states that given a computable probability space $(\mathbb{X}, \mu)$, a pair (A, B) is a Martin-Löf random element of the product space $(\mathbb{X}, \mu) \times (\mathbb{X}, \mu)$ if and only if A and B are mutually Martin-Löf random elements of $(\mathbb{X}, \mu)$. □

Theorem 5.3 *At any time $t > 0$, for $B(t)$ a planar Martin-Löf random path started at (0,0), B is not random relative to any point $(B_x(t), B_y(t))$ on the path, other than the origin.*

Proof Let z be a point of the plane. The probability that $\mathscr{B}$ passes through z is 0. Morevover, for all integers d, T, the set

$$\{B \mid \text{deficiency}(B) \leq d \wedge \exists t \leq T\, B(t) = z\}$$

is a $\Pi^0_1(z)$-class. Indeed because the set of B of randomness deficiency at most d is a Π^0_1 class, and knowing a bound on the randomness deficiency of B, one can approximate $\{B(t) \mid t \in [0, T]\}$ with arbitrary precision. Thus, the predicate $[\exists t \leq T\, B(t) = z]$ means that for all ε, the distance between B_ε (an effectively computed approximation of B at distance at most ε) is at distance at most, say, 2ε of z which is clearly a $\Pi^0_1(z)$ sentence. The above class being $\Pi^0_1(z)$ and having measure 0, no z-Martin-Löf random (in fact, no z-Kurtz random) can belong to it. □

Corollary 5.4 *For B a Martin-Löf random planar path, the range of B has zero area.*

Proof Only Lebesgue measure zero many points derandomize any particular real, so any Martin-Löf random path hits only Lebesgue measure zero many points. □

Corollary 5.5 *For any point $(x, y) \neq (0, 0)$, only measure zero many Brownian paths hit (x, y). (Almost surely, Brownian motion does not hit a given point.)*

Proof A real derandomizes only Lebesgue measure zero many reals. □

Corollary 5.6 *At any time $t > 0$, for $B(t)$ a standard planar Martin-Löf random Brownian motion, B does not pass through any computable point.*

Proof A Martin-Löf random path is always random relative to a computable point. □

5.2 Dirichlet Problem

The Dirichlet problem asks the following question: given a domain (ie connected open set) $U \subseteq \mathbb{R}^n$ and a function ϕ defined everywhere on the boundary ∂U of U, is there a unique, continuous function u such that u is harmonic on the interior of U and $u = \phi$ on ∂U? The Dirichlet problem arises whenever one considers notions of potential – for example, the problem may be thought of as finding the temperature of the interior of a heat-conducting region for which the temperature on the boundary is known, or alternatively, finding the electric potential on the interior of a region for which the charge on the boundary is known.

These physical interpretations of the problem make it clear that there should be a unique solution, and indeed, many ways of finding this unique solution are known. One method of solving the Dirichlet problem which arises from an intuition of heat diffusion in a heat-conducting substance uses the mathematical model of Brownian motion.

Definition 5.7 A point $x \in \mathbb{R}^n$ is called *regular* for a closed set $C \subset \mathbb{R}^n$ if a Brownian motion started at x does not immediately leave C with positive probability, ie if the the following holds

$$\mathbb{P}_x \left[\inf\{t > 0 : \mathscr{B}(t) \in C\} = 0\right] = 1.$$

A useful criterion for regularity is the so-called *Poincaré cone condition*: if there exists if there exists a cone V based at x with positive opening angle and $h > 0$ such that $V \cap \beta(x, h) \subset C$, where $\beta(x, h)$ denotes the open ball around x of radius h, then x is regular for C.

Theorem 5.8 (see Mörters and Peres [20, Theorem 8.5]) *Suppose $U \subset \mathbb{R}^d$ is a bounded domain such that every boundary point is regular for U^c, and suppose ϕ is a continuous function on ∂U. Let $\tau = \inf\{t > 0 : \mathscr{B}(t) \in \partial U\}$, which is an almost surely finite stopping time. Then the function $u : \overline{U} \to \mathbb{R}$ given by*

$$u(x) = \mathbb{E}_x \left[\phi(\mathscr{B}(\tau))\right], \quad \text{for } x \in \overline{U},$$

is the unique continuous function on $\overline{U}$ which is harmonic on U and such that $u(x) = \phi(x)$ for all $x \in \partial U$.

The central result of this section is that the natural effective version of this theorem holds. Namely, we prove the following.

Theorem 5.9 *Let $U \subset \mathbb{R}^d$ be a bounded domain such that every boundary point is regular for U^c. Assume that both U and $\overline{U}^c$ are effectively open and that ϕ is a computable function on ∂U. Then the function u of Theorem 5.8 is computable (and is the unique continuous function on $\overline{U}$ which is harmonic on U and such that $u(x) = \phi(x)$ for all $x \in \partial U$).*

This result is more general than the result of Andreev et al [1] who essentially show this result in the case where the boundary of U is a single (computable) Jordan curve. This is incomparable with our theorem, since some Jordan curves have irregular boundary points, but our theorem covers different cases such as an annulus in dimension 2 of computable center and radii. Bridges and McKubre-Jordens [4] also studied similar questions, but in a slightly different context, namely in the Bishop-style constructive mathematical framework BISH.

It is not immediately clear in the above theorem what it means for a function ϕ defined on ∂U to be computable. However, under the assumptions of the theorem, ∂U must be a recursive closed set (refer to Section 3.1 for the definition of recursive closed set), and thus ∂U can be viewed as a computable metric subspace of $\mathbb{R}^d$, on which there is a canonical notion of computable function. Let us explain why the hypotheses of the theorem imply that ∂U is recursive. First, notice that an open ball β intersects ∂U if and only if it intersects both U and $\overline{U}^c$. Indeed, if it intersected both but not $\partial U = \overline{U} \cap U^c$, then $\beta \cap U$ and $\beta \cap \overline{U}^c$ – which are both open – would form a partition of β, which would contradict the connectedness of β. Conversely if β intersects ∂U, then it must intersect U since every point of ∂U is the limit of points of U, and it must intersect $\overline{U}^c$ due to the Poincare cone condition. If β is a rational open ball, $\beta \cap U \neq \emptyset$ and $\beta \cap \overline{U}^c \neq \emptyset$ are Σ^0_1 predicates (since U and $\overline{U}^c$ are effectively open), and thus so is $\beta \cap \partial U \neq \emptyset$. Finally, a closed ball $\overline{\beta}$ is disjoint from ∂U, then it must be at positive distance of it, and thus must be contained in an open ball β' disjoint from ∂U. By the above discussion, this β' must be either contained in U, or contained in $\overline{U}^c$. It remains to notice that $\exists \beta' \supseteq \overline{\beta} : \beta' \subseteq U \vee \beta' \subseteq \overline{U}^c$ (where β' ranges over rational open balls), is a Σ^0_1 predicate.

The core of the proof of Theorem 5.9 is that the first hitting time of ∂U of a path B started inside $\overline{U}$ is layerwise computable in B, uniformly in the starting point, which is a generalization of Corollary 3.7.

Proposition 5.10 *Let U be a domain satisfying the condition of Theorem 5.9, and let $x \in \overline{U}$. For B a Martin-Löf random (relative to x) path, $\tau(x + B) = \inf\{t > 0 : (x + B(t)) \in \partial U\}$ is finite and is computable layerwise in B and uniformly in x.*

Proof Fix x in $\overline{U}$. Let us first check that $\tau(x+B)$ is defined (ie is finite) for every B Martin-Löf random relative to x. By the hypothesis, U is bounded, say of diameter D. There exists an l such that the probability for a Brownian motion to move by a distance at least D during a time interval of length l is greater than $1/2$. Thus, if $\tau(x+B)$ is infinite, this means that $x+B$ fails the test $(\mathcal{V}_n)_{n\in\mathbb{N}}$, where $\mathcal{V}_n$ is the set of Brownian paths which do not move by more than D on each of the intervals $[kl,(k+1)l]$ with $k=0,...,n$. By the Markov property we have $\mathbb{P}(\mathcal{V}_n)\le 2^{-n}$ and it is easy to see that $\mathcal{V}_n$ is an effectively open set uniformly in n. Thus $x+B$ is not Martin-Löf random, which by the shift-invariance of $\mathbb{P}$ shows that B is not Martin-Löf random relative to x. Note that this argument further shows that given B and a bound on the randomness deficiency of B, we can find a T such that $\{B(s)\mid s\in[0,T]\}$ is guaranteed to intersect ∂U.

We know that $\mathbb{P}$-almost surely, for each $\delta>0$, there is a time $t\in[\tau(x+\mathscr{B}),\tau(x+\mathscr{B})+\delta]$ such that $(x+\mathscr{B}(t))\in\overline{U}^c$ (this is what the regularity condition ensures). This means that $\mathbb{P}$-almost surely, the stopping time $\tau(x+\mathscr{B})$ coincides with the infimum over times t such that $(x+\mathscr{B}(t))\in\overline{U}^c$. Call τ' this infimum (which is a random variable).

Now observe that $\tau(x{+}B)$ is lower semi-computable layerwise in B and uniformly in x. Indeed, the predicate $t<\tau$ means that $\{x+B(s)\mid s\in[0,t]\}$ does not intersect ∂U, which means that there is some $\varepsilon>0$ such that $\{x+B(s)\mid s\in[0,t]\}$ is at distance at least ε from ∂U, and thus contained in U (since $x+B$ starts at $x\in\overline{U}$). By the hypothesis, U is effectively open, thus one can enumerate the rational open balls contained in U. But we know that given x, B, a bound on its randomness deficiency, and t, we can approximate the curve $\{x+B(s)\mid s\in[0,t]\}$ to arbitrary precision, and thus eventually see that it is covered by the enumeration of open balls contained in U.

On the other hand, $\tau'(x+B)$ is upper semi-computable layerwise in B, because the set $\{t\mid x+\mathscr{B}(t)\in\overline{U}^c\}$ is effectively open layerwise in B and uniformly in x (again due to the fact that we can approximate B to arbitrary precision knowing a bound on its randomness deficiency, and also due to the fact that $\overline{U}^c$ is effectively open) and the infimum of a Σ^0_1 set of reals is upper semi-computable.

As we have seen, the probability that $\tau<\tau'$ is 0. Since $\tau(x+B)$ is lower semi-computable layerwise in B uniformly in x and $\tau'(x+B)$ is upper semi-computable layerwise in B and uniformly in x, the distance $\tau'(x+B)-\tau(x+B)$ is upper semi-computable layerwise in B and uniformly in x. Suppose for the sake of contradiction that B is Martin-Löf random relative to x and $\tau'(x+B)-\tau(x+B)>q$ for some positive rational q. Let d be the randomness deficiency of B. Then B belongs to the class of functions f such that the x-deficiency of f is less or equal to d and

$\tau'(x+f) - \tau(x+f) \geq q$. This is a $\Pi^{0,x}_1$ class (by layerwise upper semi-computability of the difference $\tau' - \tau$) of $\mathbb{P}$-measure 0, thus B is not even Kurtz random relative to x, a contradiction.

We have established that $\tau(x+B) = \tau'(x+B)$ for every B which is Martin-Löf random relative to x. By the above, this means that the common value is both upper and lower semi-computable layerwise in B and uniformly in x, thus computable layerwise in B and uniformly in x. □

Theorem 5.9 immediately follows from this last proposition. Indeed, as the function $(x, B) \mapsto \tau(x + B)$ is computable in x and layerwise in B, so is its composition with the computable function ϕ, and by Theorem 1.2 (observing that ϕ, being continuous, must be bounded on the compact set ∂U) so is $x \mapsto \int_B \phi(\tau(x + B))\, d\mathbb{P}$, which is what we wanted to prove.

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