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Peano and Osgood theorems via effective infinitesimals

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Abstract: We provide choiceless proofs using infinitesimals of the global versions of Peano's existence theorem and Osgood's theorem on maximal solutions. We characterize all solutions in terms of infinitesimal perturbations. Our proofs are more effective than traditional non-infinitesimal proofs found in the literature. The background logical structure is the internal set theory **SPOT**, conservative over **ZF**.

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1 Introduction

Nonstandard analysis (NSA) is sometimes criticized for its implicit dependence on strong forms of the Axiom of Choice (**AC**). Indeed, if $*$ is the mapping that assigns to each $X \subseteq \mathbb{N}$ its nonstandard extension $^*X$, and if $\nu \in {}^*\mathbb{N} \setminus \mathbb{N}$ is an unlimited integer, then the set $U = \{X \subseteq \mathbb{N} \mid \nu \in {}^*X\}$ is a nonprincipal ultrafilter over $\mathbb{N}$. Of course strong forms of **AC**, such as Zorn's Lemma, are a staple of modern set-theoretic mathematics, but it is undesirable to have to rely on them for results in ordinary mathematics dealing with Calculus or differential equations (see Simpson [18] for a discussion of the distinction between *set-theoretic* and *ordinary* mathematics). The traditional proofs of most theorems in ordinary mathematics are *effective*: they do not use **AC**.[1] A few results, such as the equivalence of the ε-δ definition and the sequential definition of continuity for functions $f\colon \mathbb{R} \to \mathbb{R}$, require weak forms of **AC**, notably the Axiom of Countable Choice (**ACC**) or the stronger Axiom of Dependent Choice (**ADC**). These weak forms are generally accepted in ordinary mathematics; they do not imply the strong consequences of **AC** such as the existence of nonprincipal ultrafilters

[1] In this paper the word *effective* means *without the Axiom of Choice*. In reverse mathematics, constructive mathematics and other areas, it usually has more restrictive meaning.

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or the Banach–Tarski paradox (see Jech [12], Howard and Rubin [7]). We refer to such proofs as *semi-effective*.

An answer to the above criticism of NSA is offered by recent developments in the axiomatic/syntactic approach that dates back to the work of Hrbacek [8] and Nelson [15]. A number of axiomatic systems for NSA have been proposed, of which Nelson's **IST** is the best known. We refer to Kanovei and Reeken's monograph [13] for a comprehensive discussion of such axiomatic frameworks. An accessible introduction to **IST** is Robert [16].

The theory **IST** includes the axioms of **ZFC**, so one could ask whether the dependence on **AC** could be avoided by deleting **AC** from the axioms constituting **IST**. It turns out that in the resulting theory one can still prove the existence of nonprincipal ultrafilters, by an argument similar to the one given above for the model-theoretic approach (see Hrbacek [9] and the paragraph following Lemma 2.5 below).

In Hrbacek and Katz [10] the authors have developed an axiomatic system for NSA with the acronym **SPOT**, a subtheory of **IST**. The theory **SPOT** is a conservative extension of **ZF**. This means that every statement in the $\in$–language provable in **SPOT** is provable already in **ZF**. In particular, **AC** and the existence of nonprincipal ultrafilters are not provable is **SPOT**, because they are not provable in **ZF**. A stronger theory **SCOT** which is a conservative extension of **ZF** + **ADC** is also considered there. Hence proofs in **SPOT** are effective, and proofs in **SCOT** are semi-effective.

Some examples of constructions in nonstandard analysis formalized in these theories are given in [10]. In particular, it is shown there how the Riemann integral can be defined in **SPOT** using partitions into infinitesimal subintervals, and the countably additive Lebesgue measure in **SCOT** using counting measures. The expository article Hrbacek and Katz [11] presents in **SCOT** various nonstandard arguments related to compact sets and continuity.

In Section 2 we state the axioms of **SPOT**, list some of their consequences, and prove a stronger version of the Standard Part principle **SP** that is crucial in the preliminary Section 3.

In Sections 4 - 6 we give nonstandard proofs in **SPOT** of the global versions of Peano's and Osgood's theorems concerning the existence of solutions of ordinary differential equations. While the nonstandard approach using Euler approximations with an infinitesimal step that we employ is well known for local solutions (see, eg, Albeverio, Høegh-Krohn, Fenstad, and Lindstrøm [1, page 30]), we offer three innovations:

- The axiomatic system **SPOT** enables us to use infinitesimal methods without the underlying assumption of the existence of nonprincipal ultrafilters or any other strong form of **AC**.
- We construct global, ie, noncontinuable, solutions rather than local solutions.
- Traditional proofs of the existence of noncontinuable solutions typically depend on **ADC**; see Remark 7.1. By contrast, our proof does not assume any form of **AC** at all.

We first prove (Theorem 4.1) that every infinitesimal perturbation ε determines a unique global solution y_ε (some or all of these solutions may be the same). We next prove (Lemma 5.1) that every solution that is not global is a restriction of some y_ε. Hence every solution is either global or can be extended to a global one (Corollary 5.2) and every global solution is of the form y_ε for some infinitesimal perturbation ε (Theorem 5.3). Finally we state the global Osgood's theorem (Theorem 6.2). The proof shows first that there is a local maximal solution (Lemma 6.5 and the last part of the sentence that precedes it). The last paragraph of the proof obtains the global maximal solution as the union of all local ones.

2 Theory SPOT

By an $\in$–language we mean the language that contains a binary membership predicate $\in$ and is enriched by defined symbols for constants, relations, functions and operations customary in traditional mathematics. For example, it contains names $\mathbb{N}$ and $\mathbb{R}$ for the sets of natural and real numbers; they are viewed as defined in the traditional way. ($\mathbb{N}$ is the least inductive set, $\mathbb{R}$ is defined in terms of Dedekind cuts or Cauchy sequences.) The symbols $<, +$ and $\times$ denote the ordering, addition and multiplication of real numbers, and so on without further explanation. The classical theories **ZF** and **ZFC** are formulated in the $\in$–language.

The language of **SPOT** contains an additional unary predicate **st**. **SPOT** is a subtheory of **IST** and its bounded version **BST** (see [13]). We use $\forall$ and $\exists$ as quantifiers over sets and $\forall^{\mathbf{st}}$ and $\exists^{\mathbf{st}}$ as quantifiers over standard sets. The theory **SPOT** has the following axioms.

ZF (Zermelo - Fraenkel Set Theory)

T (Transfer) Let ϕ be an $\in$–formula with standard parameters. Then:

$$\forall^{\mathbf{st}} x\ \phi(x) \rightarrow \forall x\ \phi(x)$$

O (Nontriviality) $\exists \nu \in \mathbb{N}\ \forall^{\mathbf{st}} n \in \mathbb{N}\ (n \neq \nu)$

SP′ (Standard Part)

$$\forall A \subseteq \mathbb{N}\ \exists^{\mathbf{st}} B \subseteq \mathbb{N}\ \forall^{\mathbf{st}} n \in \mathbb{N}\ (n \in B \longleftrightarrow n \in A)$$

The theory **SPOT** proves the following results (see [10]).

Lemma 2.1 *Standard natural numbers precede all nonstandard ones:*

$$\forall^{\mathbf{st}} n \in \mathbb{N}\ \forall m \in \mathbb{N}\ (m < n \rightarrow \mathbf{st}(m))$$

Note that $\{0, 1, \ldots, n-1\}$ is a finite set for every $n \in \mathbb{N}$; it is nonstandard when n is nonstandard.

Lemma 2.2 (Countable Idealization) *Let ϕ be an $\in$–formula with arbitrary parameters.*

$$\forall^{\mathbf{st}} n \in \mathbb{N}\ \exists x\ \forall m \in \mathbb{N}\ (m \leq n\ \rightarrow \phi(m, x)) \longleftrightarrow \exists x\ \forall^{\mathbf{st}} n \in \mathbb{N}\ \phi(n, x)$$

The dual form of Countable Idealization is:

$$\exists^{\mathbf{st}} n \in \mathbb{N}\ \forall x\ \exists m \in \mathbb{N}\ (m \leq n\ \wedge\ \phi(m, x)) \longleftrightarrow \forall x\ \exists^{\mathbf{st}} n \in \mathbb{N}\ \phi(n, x)$$

Countable Idealization easily implies the following more familiar form. We use $\forall^{\mathbf{st\,fin}}$ and $\exists^{\mathbf{st\,fin}}$ as quantifiers over standard finite sets.

Corollary 2.3 *Let ϕ be an $\in$–formula with arbitrary parameters. For every standard countable set A:*

$$\forall^{\mathbf{st\,fin}} a \subseteq A\ \exists x\, \forall y \in a\ \phi(x, y) \longleftrightarrow \exists x\, \forall^{\mathbf{st}} y \in A\ \phi(x, y)$$

The axiom **SP′** is often stated and used in the form

$$(\mathbf{SP}) \qquad \forall x \in \mathbb{R}\ \big(x \text{ limited } \rightarrow \exists^{\mathbf{st}} r \in \mathbb{R}\ (x \approx r)\big)$$

where x is *limited* iff $|x| \leq n$ for some standard $n \in \mathbb{N}$, and $x \approx r$ iff $|x - r| \leq 1/n$ for all standard $n \in \mathbb{N}$, $n \neq 0$; x is *infinitesimal* if $x \approx 0\ \wedge\ x \neq 0$. The unique standard real number r in **SP** is called the *standard part of* x or the *shadow of* x; notation $r = \mathbf{sh}(x)$.

We have the following equivalence.

Lemma 2.4 *The statements* **SP′** *and* **SP** *are equivalent (over the theory* $\mathbf{ZF} + \mathbf{O} + \mathbf{T}$*).*

$\mathbf{SP'}$ can also be reformulated as an axiom schema (*Countable Standardization for $\in$–formulas*):

($\mathbf{SP''}$) Let ϕ be an $\in$–formula with arbitrary parameters. Then
$$\exists^{\mathbf{st}} S\, \forall^{\mathbf{st}} n\ (n \in S \longleftrightarrow n \in \mathbb{N} \wedge \phi(n)).$$

Lemma 2.5 *The statement $\mathbf{SP'}$ and the schema $\mathbf{SP''}$ are equivalent (over the theory $\mathbf{ZF}+\mathbf{O}+\mathbf{T}$).*

Proof Apply $\mathbf{SP'}$ to the set $A = \{n \in \mathbb{N} \mid \phi(n)\}$ (A exists because ϕ is an $\in$–formula). □

Standardization in full strength, as postulated in **IST**, **BST**, etc., implies the existence of nonprincipal ultrafilters over $\mathbb{N}$: take a nonstandard $\nu \in \mathbb{N}$ and let U be the standard subset of $\mathcal{P}(\mathbb{N})$ such that $\forall^{\mathbf{st}} X \subseteq \mathbb{N}\ (X \in U \longleftrightarrow \nu \in X)$. Nonetheless, two important special cases of Standardization can be proved in **SPOT**.

The scope of Countable Standardization can be expanded to a larger class of formulas.

Definition 2.6 An $\mathbf{st}$-$\in$–formula $\Phi(v_1, \dots, v_r)$ is $\mathbf{st}$-*prenex* if it is of the form
$$\mathsf{Q}^{\mathbf{st}} u_1 \dots \mathsf{Q}^{\mathbf{st}} u_s\, \psi(u_1, \dots, u_s, v_1, \dots, v_r)$$
where ψ is an $\in$–formula and each Q stands for $\exists$ or $\forall$.

In other words, all occurrences of $\forall^{\mathbf{st}}$ or $\exists^{\mathbf{st}}$ in Φ appear before all occurrences of $\forall$ or $\exists$.

We use $\forall^{\mathbf{st}}_{\mathbb{N}} u \dots$ and $\exists^{\mathbf{st}}_{\mathbb{N}} u \dots$ as quantifiers over standard natural numbers, ie as shorthand for (respectively) $\forall u\, (u \in \mathbb{N} \wedge \mathbf{st}(u) \rightarrow \dots)$ and $\exists u\, (u \in \mathbb{N} \wedge \mathbf{st}(u) \wedge \dots)$.

An $\mathbf{st}_{\mathbb{N}}$-*prenex* formula is a formula of the form
$$\mathsf{Q}^{\mathbf{st}}_{\mathbb{N}} u_1 \dots \mathsf{Q}^{\mathbf{st}}_{\mathbb{N}} u_s\, \psi(u_1, \dots u_s, v_1, \dots, v_r)$$
where ψ is an $\in$–formula.

The theory **SPOT** proves the following stronger version of Countable Standardization that is used repeatedly in this paper.

Proposition 2.7 (Countable Standardization for $\mathbf{st}_{\mathbb{N}}$–prenex formulas) *Let Φ be an $\mathbf{st}_{\mathbb{N}}$–prenex formula with arbitrary parameters. Then:*
$$\exists^{\mathbf{st}} S\, \forall^{\mathbf{st}} n\ \ (n \in S \longleftrightarrow n \in \mathbb{N} \wedge \Phi(n))$$

Of course, $\mathbb{N}$ can be replaced by any standard countable set.

Proof We give the argument for a typical case

$$\forall^{\mathbf{st}}_{\mathbb{N}} u_1 \, \exists^{\mathbf{st}}_{\mathbb{N}} u_2 \, \forall^{\mathbf{st}}_{\mathbb{N}} u_3 \; \psi(u_1, u_2, u_3, v).$$

By $\mathbf{SP}''$ there is a standard set R such that for all standard n_1, n_2, n_3, n

$$\langle n_1, n_2, n_3, n \rangle \in R \longleftrightarrow \langle n_1, n_2, n_3, n \rangle \in \mathbb{N}^4 \,\wedge\, \psi(n_1, n_2, n_3, n).$$

We let $R_{n_1,n_2,n_3} = \{n \in \mathbb{N} \mid \langle n_1, n_2, n_3, n \rangle \in R\}$ and

$$S = \bigcap_{n_1 \in \mathbb{N}} \bigcup_{n_2 \in \mathbb{N}} \bigcap_{n_3 \in \mathbb{N}} R_{n_1,n_2,n_3}.$$

Then S is standard and for all standard n:

$$\begin{aligned} n \in S &\longleftrightarrow \forall n_1 \in \mathbb{N} \, \exists n_2 \in \mathbb{N} \, \forall n_3 \in \mathbb{N} \; (n \in R_{n_1,n_2,n_3}) \\ &\longleftrightarrow \text{(by Transfer)} \; \forall^{\mathbf{st}}_{\mathbb{N}} n_1 \, \exists^{\mathbf{st}}_{\mathbb{N}} n_2 \, \forall^{\mathbf{st}}_{\mathbb{N}} n_3 \; (n \in R_{n_1,n_2,n_3}) \\ &\longleftrightarrow \text{(by definition of } R) \; \forall^{\mathbf{st}}_{\mathbb{N}} n_1 \, \exists^{\mathbf{st}}_{\mathbb{N}} n_2 \, \forall^{\mathbf{st}}_{\mathbb{N}} n_3 \; \psi(n_1, n_2, n_3, n) \\ &\longleftrightarrow \Phi(n) \end{aligned}$$

□

The second special case of Standardization involves **st**–prenex formulas with only the standard parameters.

Lemma 2.8 *Let* $\Phi(v_1, \ldots, v_r)$ *be an* **st***–prenex formula with standard parameters. Then* $\forall^{\mathbf{st}} S \, \exists^{\mathbf{st}} P \, \forall^{\mathbf{st}} v_1, \ldots, v_r$

$$\langle v_1, \ldots, v_r \rangle \in P \longleftrightarrow \langle v_1, \ldots, v_r \rangle \in S \,\wedge\, \Phi(v_1, \ldots, v_r).$$

Proof Let $\Phi(v_1 \ldots, v_r)$ be $\mathsf{Q}^{\mathbf{st}}_1 u_1 \ldots \mathsf{Q}^{\mathbf{st}}_s u_s \, \psi(u_1, \ldots, u_s, v_1, \ldots, v_r)$ and let $\phi(v_1 \ldots, v_r)$ be $\mathsf{Q}_1 u_1 \ldots \mathsf{Q}_s u_s \, \psi(u_1, \ldots, u_s, v_1, \ldots, v_r)$. Since Φ has standard parameters, the equivalence $\Phi(v_1 \ldots, v_r) \longleftrightarrow \phi(v_1 \ldots, v_r)$ holds for all standard $v_1 \ldots, v_r$ by the Transfer principle.

The set $P = \{\langle v_1, \ldots, v_r \rangle \in S \mid \phi(v_1, \ldots, v_r)\}$ exists by the Separation principle of **ZF**, it is standard, and has the required property. □

Remark 2.9 This result has twofold importance:

(1) The meaning of every predicate that for standard inputs is defined by an **st**–prenex formula $\mathsf{Q}^{\mathbf{st}}_1 u_1 \ldots \mathsf{Q}^{\mathbf{st}}_s u_s \, \psi$ with standard parameters is automatically extended to all inputs, where it is given by the $\in$–formula $\mathsf{Q}_1 u_1 \ldots \mathsf{Q}_s u_s \, \psi$.

(2) Standardization holds for all $\in$–formulas with additional predicate symbols, as long as all these additional predicates are defined by **st**–prenex formulas with standard parameters.

3 Two examples

Formulas that occur in practice are usually not in the **st**–prenex form, but they can often be converted to it using Countable Idealization.

Definition 3.1 (Integral of continuous functions) We fix a positive infinitesimal h and the corresponding "hyperfinite line" $\{x_i \mid i \in \mathbb{Z}\}$ where $x_i = i \cdot h$. Let f be a standard real-valued function continuous on the standard interval $[a,b]$. Let i_a, i_b be such that $i_a \cdot h - h < a \leq i_a \cdot h$ and $i_b \cdot h < b \leq i_b \cdot h + h$. We define:

$$\int_a^b f(x)\,dx = \mathbf{sh}\left(\Sigma_{i=i_a}^{i_b} f(x_i) \cdot h\right) \tag{3–1}$$

It is easy to show that the value of the integral does not depend on the choice of h.

Lemma 3.2 *There is an* $\mathbf{st}_{\mathbb{N}}$*–prenex formula* $\Phi(v_1, v_2, v_3, v_4)$ *such that* $\int_a^b f(x)\,dx = r \longleftrightarrow \Phi(f, a, b, r)$ *holds for all standard* f, a, b, r*.*

Proof For standard f, a, b, r we have $\int_a^b f(x)\,dx = r$ iff:

$$\forall h \left[\forall^{\mathbf{st}}_{\mathbb{N}} n \left(|h| < \tfrac{1}{n}\right) \to \forall^{\mathbf{st}}_{\mathbb{N}} m \left(|\Sigma_{i=i_a}^{i_b} f(x_i) \cdot h - r| < \tfrac{1}{m}\right)\right]$$

(It is understood that h, n, m are not 0). This expression can be rewritten as:

$$\forall h\, \forall^{\mathbf{st}}_{\mathbb{N}} m\, \exists^{\mathbf{st}}_{\mathbb{N}} n \left[|h| \geq \tfrac{1}{n} \vee |\Sigma_{i=i_a}^{i_b} f(x_i) \cdot h - r| < \tfrac{1}{m}\right]$$

We swap the outmost universal quantifiers and apply the dual version of Countable Idealization (Lemma 2.2) to get

$$\forall^{\mathbf{st}}_{\mathbb{N}} m\, \exists^{\mathbf{st}}_{\mathbb{N}} n\, \forall h\, \exists k \leq n \left[|h| \geq \tfrac{1}{k} \vee |\Sigma_{i=i_a}^{i_b} f(x_i) \cdot h - r| < \tfrac{1}{m}\right]$$

which is an $\mathbf{st}_{\mathbb{N}}$–prenex formula, clearly equivalent to

$$\forall^{\mathbf{st}}_{\mathbb{N}} m\, \exists^{\mathbf{st}}_{\mathbb{N}} n\, \forall h \left[|h| \geq \tfrac{1}{n} \vee |\Sigma_{i=i_a}^{i_b} f(x_i) \cdot h - r| < \tfrac{1}{m}\right]. \qquad \square$$

One can now use Standardization for **st**–prenex formulas with standard parameters to conclude that, for example, for every standard f, a there exists a standard function F such that $F(z) = \int_a^z f(x)\,dx$ for all standard $z \in [a,b]$. By Remark 2.9 (1), the last equation holds for all $z \in [a,b]$. Of course, the usual arguments show that the above definition of the integral agrees with the traditional ϵ-δ one for all standard f, a, b, r.

The following observation is crucial for the proof of Proposition 3.6.

Lemma 3.3 *Let w be a function,* $\operatorname{dom} w = D_w \subseteq \mathbb{R}$ *and* $\operatorname{ran} w \subseteq \mathbb{R}$*. Then the formula*

$$\Psi(x,y): \quad \exists \alpha \in D_w\ [x \approx \alpha\ \wedge\ (y \approx w(\alpha)\ \vee\ y \geq w(\alpha))]$$

is equivalent to an $\mathbf{st}_{\mathbb{N}}$*–prenex formula (with the parameter* w*).*

Proof The formula $\Psi(x,y)$ can be written as

$$\exists \alpha \in D_w\ [\forall^{\mathbf{st}}_{\mathbb{N}} i\,(|x-\alpha| < \tfrac{1}{i+1})\ \wedge\ (\forall^{\mathbf{st}}_{\mathbb{N}} j\,(|y - w(\alpha)| < \tfrac{1}{j+1})\ \vee\ y \geq w(\alpha))]$$

which is equivalent to:

$$\exists \alpha \in D_w\ \forall^{\mathbf{st}}_{\mathbb{N}} i\ \forall^{\mathbf{st}}_{\mathbb{N}} j\,[(|x-\alpha| < \tfrac{1}{i+1})\ \wedge\ (|y - w(\alpha)| < \tfrac{1}{j+1}\ \vee\ y \geq w(\alpha))]$$

This is equivalent to

$$\exists \alpha \in D_w\ \forall^{\mathbf{st}}_{\mathbb{N}} n\ \ [(|x-\alpha| < \tfrac{1}{n+1})\ \wedge\ (|y - w(\alpha)| < \tfrac{1}{n+1}\ \vee\ y \geq w(\alpha))]$$

(let $n = \min\{i,j\}$), and finally (Countable Idealization, Lemma 2.2) to the $\mathbf{st}_{\mathbb{N}}$–prenex formula:

$$\forall^{\mathbf{st}}_{\mathbb{N}} n\, \exists \alpha \in D_w\ \forall m \leq n\ [(|x-\alpha| < \tfrac{1}{m+1})\ \wedge\ (|y - w(\alpha)| < \tfrac{1}{m+1}\ \vee\ y \geq w(\alpha))]$$

The last formula of course simplifies to:

$$\forall^{\mathbf{st}}_{\mathbb{N}} n\, \exists \alpha \in D_w\ [(|x-\alpha| < \tfrac{1}{n+1})\ \wedge\ (|y - w(\alpha)| < \tfrac{1}{n+1}\ \vee\ y \geq w(\alpha))] \qquad \square$$

Definition 3.4 Let w be a function, $\operatorname{dom} w = D_w \subseteq I$ where $I \subseteq \mathbb{R}$ is a standard interval, and $\operatorname{ran} w \subseteq \mathbb{R}$.

- The function w is *densely defined on* I if for every standard $x \in I$ there is $\alpha \in D_w$ such that $\alpha \approx x$.
- The function w is (uniformly) S–*continuous* if for $\alpha, \beta \in D_w$, $\alpha \approx \beta$ implies $w(\alpha) \approx w(\beta)$.

Lemma 3.5 *A function w is S–continuous iff for every standard $\epsilon > 0$ there is a standard $\delta > 0$ such that for $\alpha, \beta \in D_w$, $|\alpha - \beta| < \delta$ implies $|w(\alpha) - w(\beta)| < \epsilon$.*

Proof The usual arguments work in **SPOT**; see eg Hrbacek and Katz [11]. □

The next proposition follows immediately from the Standardization principle of **IST** or **BST**, but to prove it in **SPOT** we need to consider an approximation to the set W on the rationals, to which we can apply Countable Standardization for $\mathbf{st}_{\mathbb{N}}$–prenex formulas.

Proposition 3.6 *If w is S–continuous and densely defined on I, then there is a standard function W such that, for all standard $x, y \in \mathbb{R}$, $\langle x, y\rangle \in W$ if and only if $x \approx \alpha$ and $y \approx w(\alpha)$ for some $\alpha \in D_w$.*

The proof of Proposition 3.6 appears below, following the proof of Lemma 3.8.

Definition 3.7 The existence of the standard set

$$Z = {}^{\mathbf{st}}\{\langle q, r\rangle \in (I \cap \mathbb{Q}) \times \mathbb{Q} \mid \exists \alpha \in D_w\,[q \approx \alpha \wedge (r \approx w(\alpha) \vee r \geq w(\alpha)]\}$$

is justified in Lemma 3.3.

For $q \in I \cap \mathbb{Q}$ let $Z_q = \{r \in \mathbb{Q} \mid \langle q, r\rangle \in Z\}$ and $W_0(q) = \inf Z_q$, if it exists (it can happen that $Z_q = \emptyset$ or $Z_q = \mathbb{Q}$, in which cases $W_0(q)$ is undefined). Finally, let W be the closure of (the graph of) W_0. We show below that the standard set W has the property from Proposition 3.6.

Lemma 3.8 *If $q \in I \cap \mathbb{Q}$ is standard, then $q \in \operatorname{dom} W_0$ if and only if there exists $\alpha \in D_w$ such that $\alpha \approx q$ and $w(\alpha)$ is limited. If this is the case, then $W_0(q) = \mathbf{sh}(w(\alpha))$.*

Proof If $\alpha, \beta \in D_w$, $q \approx \alpha$ and $q \approx \beta$, then $w(\alpha) \approx w(\beta)$, so we have $Z_q = {}^{\mathbf{st}}\{r \in \mathbb{Q} \mid r \approx w(\alpha) \vee r \geq w(\alpha)\}$, independently of the choice of α. If $w(\alpha)$ is limited, then $\inf Z_q = \mathbf{sh}(w(\alpha))$. If $w(\alpha)$ is unlimited, then $Z_q = \emptyset$ or $Z_q = \mathbb{Q}$, so $W_0(q)$ is undefined. □

Proof of Proposition 3.6 Assume that $x \approx \alpha$ and $y \approx w(\alpha)$ for $\alpha \in D_w$. Given any standard $\epsilon > 0$, take a standard $\delta > 0$ witnessing S-continuity of w, a standard $q \in \mathbb{Q} \cap I$ such that $|x - q| < \min\{\delta, \epsilon\}$ and some $\beta \approx q$, $\beta \in D_w$. Then $|\alpha - \beta| < \delta$, and hence $|w(\alpha) - w(\beta)| < \epsilon$. It follows that $w(\beta)$ is limited. By Lemma 3.8, $w(\beta) \approx W_0(q)$, so $|x - q| < \epsilon$ and $|y - W_0(q)| < \epsilon$. This shows that $\langle x, y\rangle \in W$.

Conversely, if $\langle x, y\rangle \in W$, then for every standard $\epsilon > 0$ there is $q \in \operatorname{dom} W_0$ such that $|x - q| < \epsilon$ and $|y - W_0(q)| < \epsilon$. Let $\alpha \in D_w$, $\alpha \approx q$; then $w(\alpha) \approx W_0(q)$, $|x - \alpha| < \epsilon$ and $|y - w(\alpha)| < \epsilon$. By Countable Idealization (Lemma 2.2) there is $\alpha \in D_w$ such that for all standard $\epsilon > 0$ we have $|x - \alpha| < \epsilon$ and $|y - w(\alpha)| < \epsilon$. Then $x \approx \alpha$ and $y \approx w(\alpha)$. □

4 Peano's Existence Theorem in SPOT

Theorem 4.1 (Global Peano's Theorem) *Let $F : [0, \infty) \times \mathbb{R} \to \mathbb{R}$ be a continuous function. There is an interval $[0, a)$ with $0 < a \leq \infty$ and a function $y : [0, a) \to \mathbb{R}$ such that*

$$(*) \qquad y(0) = 0, \qquad y'(x) = F(x, y(x))$$

holds for all $x \in [0, a)$, and if $a \in \mathbb{R}$ then $\lim_{x \to a^-} y(x) = \pm\infty$.

Here and elsewhere, if $c \in \mathbb{R}$ is an endpoint of an interval $I = \operatorname{dom} y$, $y'(c)$ is the appropriate one-sided derivative of y at c. We call a solution of the initial value problem ($*$) that cannot be continued to any interval $[0, a')$ with $a' > a$ a *global* solution.

We generalize the familiar construction of Euler approximations with an infinitesimal step by allowing infinitesimal perturbations. This is a variation on an idea in Birkeland and Normann [2] (the main difference being that we perturb the construction of the solution, while Birkeland and Normann perturb the function F).

We will prove the theorem for standard F; the stated result follows by Transfer. The construction proceeds as follows.

Let N be a positive unlimited integer and $h = 1/N$. We fix $x_0 \geq 0$, $x_0 \approx 0$, $y_0 \approx 0$, and let $x_k = x_0 + k \cdot h$ for $k = 0, \ldots, N^2$.

Definition 4.2 An *infinitesimal perturbation* is a sequence $\boldsymbol{\varepsilon} = \langle \varepsilon_k \mid k < N^2 \rangle$ such that each $\varepsilon_k \approx 0$; we let $\varepsilon = \max\{|\varepsilon_k| \mid k < N^2\}$.

The concept is not needed for the proof of Theorem 4.1, where the simplest choice $\varepsilon_k = 0$ for all k suffices, but it is used for its generalization in Section 5.

We define y_k recursively:

$$y_{k+1} = y_k + (F(x_k, y_k) + \varepsilon_k) \cdot h \quad \text{ for } k < N^2$$

Observe that:

$$y_\ell = y_k + \sum_{i=k}^{\ell-1} (F(x_i, y_i) + \varepsilon_i) \cdot h, \quad \text{ for any } k < \ell \leq N^2$$

We next define:

$$(**) \qquad Y = {}^{\mathbf{st}}\{\langle x, y \rangle \in [0, \infty) \times \mathbb{R} \mid x \approx x_k \wedge y \approx y_k \text{ for some } k < N^2\}$$

The existence of Y in **SPOT** follows from Proposition 3.6 (let $I = [0, \infty)$ and $w(x_k) = y_k$ for $0 \leq k < N^2$). The strategy for the rest of the proof is to show that Y

is a (graph of) a continuous function defined on an open subset of $[0,\infty)$, and the restriction y of Y to the connected component of its domain containing 0 has the required properties.

Lemma 4.3 *Let $\langle x,y\rangle \in [0,\infty)\times\mathbb{R}$ be standard and $x_p - h < x \le x_p$, $y \approx y_p$ for some $p < N^2$. There exist standard $d,e,M > 0$ such that $y_k \in [y-d, y+d]$ for all $x_k \in [x, x+e)$ and $|y_k - y_\ell| \le (M+\varepsilon)\cdot|x_k - x_\ell|$ for all $x_k, x_\ell \in [x, x+e)$. In particular, if $x_k \approx x_\ell \approx x$ then $y_k \approx y_\ell$.*

If $x > 0$ then $[x, x+e)$ can be replaced by $(x-e, x+e)$.

Proof By continuity of F at $\langle x,y\rangle$ there exist standard $c,d,M > 0$ such that $|F(t,s)| \le M$ holds for all $\langle t,s\rangle \in [x,x+c]\times[y-d,y+d]$; if $x > 0$,we can assume also $c \le x$. Fix a standard e such that $0 < e < \min\{c,\ d/(M+1)\}$.

We prove by induction on k that

$$k \ge p \ \wedge\ x_k < x+e \to |y_k - y_p| \le (M+\varepsilon)\cdot|x_k - x_p|.$$

The case $k = p$ is clear. If the claim is true for k and $x_{k+1} < x+e$, we have $|y_k - y_p| \le (M+\varepsilon)\cdot|x_k - x_p| < (M+1)\cdot e \le d$ and hence the point $\langle x_k, y_k\rangle \in [x,x+c]\times[y-d,y+d]$. Now $|F(x_k,y_k)| \le M$, so $|y_{k+1} - y_k| \le (|F(x_k,y_k)| + |\varepsilon_k|)\cdot h \le (M+\varepsilon)\cdot h$ and:

$$\begin{aligned} |y_{k+1} - y_p| &\le |y_{k+1} - y_k| + |y_k - y_p| \\ &\le (M+\varepsilon)\cdot h + (M+\varepsilon)\cdot|x_k - x_p| \\ &= (M+\varepsilon)\cdot|x_{k+1} - x_p| \end{aligned}$$

Finally, $|y_\ell - y_k| \le \sum_{i=k}^{\ell-1}(|F(x_i,y_i)| + \varepsilon)\cdot h \le (M+\varepsilon)\cdot|x_\ell - x_k|$.

If $x > 0$, a symmetric "backward" argument shows that the statement holds also on the interval $(x-e, x]$. □

It follows easily that Y as in (**) is the graph of a real function. If $\langle x,y_1\rangle$, $\langle x,y_2\rangle \in Y$ are standard, then there are k and ℓ such that $\langle x,y_1\rangle \approx \langle x_k,y_k\rangle$ and $\langle x,y_2\rangle \approx \langle x_\ell,y_\ell\rangle$. Then $x_k \approx x_\ell \approx x$ and $y_1 \approx y_k \approx y_\ell \approx y_2$ and we conclude that $y_1 = y_2$. Hence Y is the graph of a function, by Transfer. From now on we write $Y(x)$ for the value of Y at $x \in \operatorname{dom} Y$.

Lemma 4.4 *The domain of the function Y is an open subset of $[0,\infty)$ containing 0, Y is continuous on $\operatorname{dom} Y$, and $Y'(x) = F(x, Y(x))$ holds for $x \in \operatorname{dom} Y$.*

Proof Clearly $0 \in \operatorname{dom} Y$. If $x \in \operatorname{dom} Y$ is standard and $y = Y(x)$, Lemma 4.3 gives an interval $I = (x-e, x+e)$ (or $I = [0,e)$) such that $x_k \in I$ implies $y_k \in [y-d, y+d]$, hence y_k is limited and $Y(\mathbf{sh}(x_k)) = \mathbf{sh}(y_k)$ is defined. If $u \in I$ is standard, $u = \mathbf{sh}(x_k)$ holds for some $x_k \in I$. Hence $Y(u) \in [y-d, y+d]$ is defined for all standard $u \in I$, and by Transfer, the same holds for all $u \in I$.

For the proof of continuity at a standard $x \in \operatorname{dom} Y$ let I be as above, $\epsilon > 0$ be standard and $\delta = \epsilon/(M+1)$. If $z \in I$ is standard and $|x - z| < \delta$, then there are k and ℓ such that $x \approx x_k$ and $z \approx x_\ell$; moreover, $Y(x) \approx y_k$ and $Y(z) \approx y_\ell$. We have $|Y(x)-Y(z)| \approx |y_\ell - y_k|$ and $|y_k - y_\ell| \le (M+\varepsilon)\cdot|x_k - x_\ell|$, where $|x_\ell - x_k| \approx |x-z| < \delta$. It follows that $|Y(x) - Y(z)| \le (M+1)\cdot\delta = \epsilon$. As usual, Transfer gives continuity for all $x \in \operatorname{dom} Y$.

It remains to prove that $Y'(x) = F(x, Y(x))$ holds for $x \in \operatorname{dom} Y$. Let I be as above, $x, z \in I$ be standard and without loss of generality $x \le z$. In the notation of the previous paragraph, we have

$$Y(z) - Y(x) \approx y_\ell - y_k = \sum_{i=k}^{\ell-1} (F(x_i, y_i) + \varepsilon_i)\cdot h \tag{1}$$

and

$$\int_x^z F(t, Y(t))\,dt \approx \sum_{i=k}^{\ell-1} F(x_i, Y(x_i))\cdot h = \sum_{i=k}^{\ell-1} (F(x_i, y_i) + \delta_i)\cdot h \tag{2}$$

where $\delta_i \approx 0$ for $k \le i < \ell$. The relation $\approx$ in (2) follows from the nonstandard theory of integration (see Definition 3.1) and the fact that $F(t, Y(t))$ is continuous on I. The relation $=$ in (2) is justified as follows: Let $x^* = \mathbf{sh}(x_i)$ and $y^* = \mathbf{sh}(y_i)$; then $Y(x^*) \approx y^*$ by the definition of Y and $Y(x_i) \approx Y(x^*)$ by the continuity of Y. The continuity of F then gives $F(x_i, y_i) \approx F(x^*, y^*) \approx F(x_i, Y(x_i))$.

The formulas (1) and (2) imply $Y(z) - Y(x) \approx \int_x^z F(t, Y(t))\,dt$, hence $Y(z) - Y(x) = \int_x^z F(t, Y(t))\,dt$ as both sides are standard. By Transfer, the relationship holds for all x, $z \in I$. It remains to apply the Fundamental Theorem of Calculus. □

Let $[0, a)$, $a > 0$, be the connected component of the domain of Y containing 0.

Lemma 4.5 *The function Y satisfies* $\lim_{x\to a^-} Y(x) = \pm\infty$.

Proof We prove that for every standard $r > 0$ there is a standard $\epsilon > 0$ such that for all standard x, $a - \epsilon < x < a$ implies $|y(x)| \ge r$.

Assume that the statement is false and fix a standard $r > 0$ such that for every standard $n \in \mathbb{N}$ there is a standard $x \in (a - \frac{1}{n}, a)$ such that $Y(x) \in (-r, r)$. Hence for every standard $n \in \mathbb{N}$ there is $k < N^2$ such that $x_k \in (a - \frac{1}{n},\ a)$ and $y_k \in (-r, r)$ (take $\langle x_k, y_k \rangle \approx \langle x, Y(x) \rangle$). By Countable Idealization (Lemma 2.2), there exists $p < N^2$ such that $y_p \in (-r, r)$ and $x_p \in (a - \frac{1}{n},\ a)$ holds for all standard $n > 0$. It follows that $x_p \approx a$; we let $b = \mathbf{sh}(y_p)$. By the definition of Y then $\langle a, b \rangle \in Y$, and hence $a \in \operatorname{dom} Y$, contradicting the fact that $[0, a)$ is a connected component of the domain of Y. □

Conclusion of proof of Theorem 4.1 Let Y be the function defined by formula ($**$). The proof of Theorem 4.1 is now concluded by letting $y = Y \restriction [0, a)$. We write y_ε when it is necessary to indicate the dependence of y on the perturbation ε. □

Remark 4.6 Note that the solution y is determined by the choice of the starting point x_0, y_0 and the infinitesimal perturbation ε. Thus we can single out a particular global solution of ($*$) by fixing N and letting $x_0 = 0$, $y_0 = 0$ and $\varepsilon_k = 0$ for all $k < N^2$.

Remark 4.7 There are obvious generalizations that do not require any additional nonstandard ideas. For example, the two-sided version:

Let $F : \mathbb{R}^2 \to \mathbb{R}$ be a continuous function. For every $\langle a, b \rangle \in \mathbb{R}^2$ there is an interval $(a^-,\ a^+)$ with $-\infty \le a^- < a < a^+ \le \infty$ and a function $y : (a^-,\ a^+) \to \mathbb{R}$ such that

$$y(a) = b, \qquad y'(x) = F(x, y(x)) \quad \text{holds for all } x \in (a^-,\ a^+),$$

and if a^- and/or a^+ is in $\mathbb{R}$, then $\lim_{x \to (a^-)^+} y(x) = \pm\infty$ and/or $\lim_{x \to (a^+)^-} y(x) = \pm\infty$.

The domain $\mathbb{R}^2$ of F can be replaced by an open set $D \subseteq \mathbb{R}^2$. One obtains a solution that tends to the boundary of D, in the sense that for every compact $K \subseteq D$ there is $c < a^+$ such that $y(x) \notin K$ holds for all $c < x < a^+$, and analogously for a^-.

The method generalizes to systems of equations.

Theorem 4.8 *Let $\mathbf{F} : \mathbf{D} \to \mathbb{R}^n$ be continuous on an open set $\mathbf{D} \subseteq \mathbb{R}^{n+1}$ and $\langle 0, \mathbf{0} \rangle \in \mathbf{D}$. The initial value problem*

$$(\star) \qquad \mathbf{y}(0) = \mathbf{0}, \qquad \mathbf{y}'(x) = \mathbf{F}(x, \mathbf{y}(x))$$

has a noncontinuable solution.

Proof For $\mathbf{u} = \langle u_0, \ldots, u_{n-1} \rangle$ and $\mathbf{v} = \langle v_0, \ldots, v_{n-1} \rangle$ in $\mathbb{R}^n$ we let $\mathbf{u} \approx \mathbf{v}$ if $u_i \approx v_i$ for all $i < n$, and $\mathbf{u} \ge \mathbf{v}$ if $u_i \ge v_i$ for all $i < n$. With this understanding, the material in Section 3, and in particular Proposition 3.6, generalizes straightforwardly to functions w with $\operatorname{ran} w \subseteq \mathbb{R}^n$. One can then follow the proof of Theorem 4.1. □

5 Applications of infinitesimal perturbations

Recall (see the conclusion of the proof of Theorem 4.1) that $y_{\boldsymbol\varepsilon}$ is a standard function defined via ($**$).

Lemma 5.1 *Let F be standard. For every standard solution y of ($*$) defined on a standard interval $[0,a)$ and every standard $c < a$, $c > 0$, there is an infinitesimal perturbation $\boldsymbol\varepsilon$ such that $y(x) = y_{\boldsymbol\varepsilon}(x)$ holds for $0 \le x \le c$.*

Proof By the mean value theorem, for each k such that $x_{k+1} \le c$ there is $t \in [x_k, x_{k+1}]$ such that $y(x_{k+1}) - y(x_k) = y'(t) \cdot h$. Let t_k be the least such t (as y' is continuous, the set of t with this property is closed). Then let $\varepsilon_k = F(t_k, y(t_k)) - F(x_k, y(x_k)) = y'(t_k) - y'(x_k) \approx 0$. For $x_{k+1} > c$ let $\varepsilon_k = 0$. Let $y_0 = y(0)$; it follows that $y_k = y(x_k)$ for all k such that $x_{k+1} \le c$: assuming the claim is true for k, we have $y_{k+1} = y_k + (F(x_k, y_k) + \varepsilon_k) \cdot h = y(x_k) + F(t_k, y(t_k)) \cdot h = y(x_k) + y'(t_k) \cdot h = y(x_{k+1})$.

If $x \in [0,c]$ is standard, take $x \approx x_k$ for $x_{k+1} \le c$; then $y_{\boldsymbol\varepsilon}(x) \approx y_k = y(x_k) \approx y(x)$, so $y_{\boldsymbol\varepsilon}(x) = y(x)$. □

Corollary 5.2 *Every solution of ($*$) extends to a global solution.*

Proof Let y defined on $[0,c)$ be a standard solution of ($*$) with F standard. If y is not global, then it has a standard continuation $\tilde{y}$ to an interval $[0,a)$ with $c < a$. By Lemma 5.1 y has a continuation $y_{\boldsymbol\varepsilon}$ which is global by Theorem 4.1. By Transfer, the claim holds for all solutions y and all functions F. □

Theorem 5.3 *For every standard global solution y of ($*$) there is an infinitesimal perturbation $\boldsymbol\varepsilon$ such that $y = y_{\boldsymbol\varepsilon}$.*

Proof Assume the domain of y is a standard interval $[0,a)$ (possibly $a = +\infty$). We fix a standard strictly increasing sequence $\langle c_n \mid n \in \mathbb{N}\rangle$ such that $c_0 > 0$ and $\lim_{n\to\infty} c_n = a$. The proof of Lemma 5.1 (with $c = c_n$) justifies the following statement. For every standard $n \in \mathbb{N}$ there is $\boldsymbol\varepsilon = \langle \varepsilon_k \mid 0 \le k < N^2\rangle$ such that for all $m \le n$ and for all $k < N^2$:

$$\left(x_{k+1} \le c_m \to \varepsilon_k = y'(t_k) - y'(x_k)\right) \wedge \left(x_{k+1} > c_m \to |\varepsilon_k| < \tfrac{1}{m+1}\right).$$

By Countable Idealization (Lemma 2.2) there is $\boldsymbol\varepsilon$ such that for all standard $n \in \mathbb{N}$ and for all $k < N^2$:

$$\left(x_{k+1} \le c_n \to \varepsilon_k = y'(t_k) - y'(x_k)\right) \wedge \left(x_{k+1} > c_n \to |\varepsilon_k| < \tfrac{1}{n+1}\right)$$

It follows that $\varepsilon_k \approx 0$ for all $k < N^2$, so $\boldsymbol{\varepsilon}$ is a perturbation. As in the proof of Lemma 5.1, $y(x) = y_{\boldsymbol{\varepsilon}}(x)$ holds for every standard $x \in [0, c_n]$, for every standard $n \in \mathbb{N}$, hence $y(x) = y_{\boldsymbol{\varepsilon}}(x)$ holds for every standard $x \in [0, a)$. By Transfer, $y(x) = y_{\boldsymbol{\varepsilon}}(x)$ for all $x \in [0, a)$. □

The results of this section generalize to the system of equations ($\star$).

6 Osgood's Theorem in SPOT

Definition 6.1 A solution $\tilde{y}$ of ($*$) defined on an interval I is *maximal on I* if $\tilde{y}(x) \geq y(x)$ holds for every solution y of ($*$) and every $x \in I \cap \operatorname{dom} y$. The solution $\tilde{y}$ is *maximal* if it is global and maximal on its domain.

Theorem 6.2 (Global Osgood's Theorem) *The initial value problem* ($*$) *has a unique maximal solution.*

Proof We assume that F is standard, fix an infinitesimal $\varepsilon > 0$ and consider the initial value problem

$$(***) \qquad z(0) = 0, \qquad z'(x) = F(x, z(x)) + \varepsilon.$$

Lemma 6.3 *There exist standard $e, M > 0$ such that, for the intervals $I = [0, e]$ and $J = [-(M+1) \cdot e, (M+1) \cdot e]$, the function $F + \varepsilon$ is bounded by M on $I \times J$ and the initial value problem* ($***$) *has a solution $u : I \to J$.*

Proof of Lemma 6.3 The arguments given in the proof of Theorem 4.1 establish the following uniform result:

Given standard $c, d, M > 0$ there is a standard $e > 0$ such that for every standard G, continuous and bounded by M on $[0, c) \times [-d, d]$, there is a solution $y : [0, e] \to [-(M+1) \cdot e, (M+1) \cdot e]$ of the initial value problem $y(0) = 0$, $y'(x) = G(x, y(x))$. By Transfer, the result holds for all such functions G.

Returning to ($***$), fix standard $c, d, M_0 > 0$ so that F is bounded by M_0 on $[0, c) \times [-d, d]$. Let $G = F + \varepsilon$ and $M = M_0 + 1$. The paragraph above gives the desired solution u. □

Lemma 6.4 *Let u be the solution of the initial value problem* ($***$) *furnished by Lemma 6.3 and let $y(0) = 0$ and $y'(x) = F(x, y(x))$ for all $x \in [0, a)$. Then $u(x) \geq y(x)$ holds for all $x \in [0, \min\{e, a\})$.*

Proof of Lemma 6.4 Let

$$\alpha = \sup\{\alpha' \mid u(x) \geq y(x) \text{ holds for all } 0 \leq x \leq \alpha'\}$$

and assume $\alpha < \min\{e, a\}$. Then $u(\alpha) \geq y(\alpha)$ and $u'(\alpha) - y'(\alpha) = \varepsilon > 0$. It follows that $u(x) > y(x)$ holds on some interval $(\alpha, \alpha']$ for $\alpha' > \alpha$, a contradiction. □

We next prove the existence of a local maximal solution. We let:

$$y_m(x) = {}^{\mathbf{st}}\{\mathbf{sh}(u(x)) \mid x \in [0, e]\}$$

The existence of the standard function y_m defined on $[0, e]$ in **SPOT** follows from Proposition 3.6 (with $I = [0, e]$ and $w = u$), using the observation that u is S–continuous: $|x-z| \approx 0$ implies $|u(x)-u(z)| = |u'(t)|\cdot|x-z| = |F(t, u(t))+\varepsilon|\cdot|x-z| \leq M\cdot|x-z| \approx 0$ (where t is between $x, z \in I$).

If y is a standard solution of $(*)$, then $y_m(x) = \mathbf{sh}(u(x)) \geq \mathbf{sh}\, y(x)) = y(x)$ holds for all standard $x \in [0, \min\{e, a\})$ by Lemma 6.4, so y_m dominates all standard solutions of $(*)$.

Lemma 6.5 *The function y_m is a solution of $(*)$ on $[0, e]$.*

Proof of Lemma 6.5 To this effect it suffices to find an infinitesimal perturbation ε such that $y_m = y_\varepsilon$ on $[0, e]$.

As in the proof of Theorem 5.1, for each k with $x_{k+1} \leq e$ let t_k be the least $t \in [x_k, x_{k+1}]$ such that $u(x_{k+1})-u(x_k) = u'(t)\cdot h$. Then let $\varepsilon_k = F(t_k, u(t_k))-F(x_k, u(x_k))$; if $x_{k+1} > e$ let $\varepsilon_k = 0$.

Let $y_0 = u(0) = 0$. If $y_k = u(x_k)$, then:

$$\begin{aligned} y_{k+1} &= y_k + (F(x_k, y_k) + \varepsilon_k)\cdot h \\ &= u(x_k) + F(t_k, u(t_k))\cdot h \\ &= u(x_k) + u'(t_k)\cdot h = u(x_{k+1}) \end{aligned}$$

It follows that $y_k = u(x_k)$ for all k such that $x_{k+1} \leq e$.

We still have to show that $\varepsilon_k \approx 0$. The function u is S–continuous: $x, z \in [0, e]$ and $x \approx z$ imply:

$$|u(x) - u(z)| \leq \left| \int_z^x (F(t, u(t)) + \varepsilon)\, dt \right| \leq M\cdot|x - z| \approx 0$$

So $t_k \approx x_k$ implies $u(t_k) \approx u(x_k)$ and $F(t_k, u(t_k)) \approx F(x_k, u(x_k))$, because F is continuous at $\langle \mathbf{sh}(x_k), \mathbf{sh}(u(x_k))\rangle$.

For standard $x \in [0, e]$ take $x \approx x_k$; we have $y_m(x) = \mathbf{sh}(u(x)) = \mathbf{sh}(u(x_k)) = \mathbf{sh}(y_k) = y_\varepsilon(x)$. By Transfer, $y_m(x) = y_\varepsilon(x)$ holds for all $x \in [0, e]$. □

The above argument establishes the existence of a solution y_m which is maximal over some interval $[0,e)$. The maximal solution $y_{\max}$ is obtained as the union of all such solutions; it is defined and maximal on some interval I. It remains to prove that $I=[0,a)$ (with $0<a\leq+\infty$) and that $y_{\max}$ is global. If $I=[0,a]$ for $a\in\mathbb{R}$, we could apply the above argument to the initial value $\langle a, y_{\max}(a)\rangle$ and obtain a continuation of $y_{\max}$ that is defined and maximal on a larger interval. Similarly, if $y_{\max}$ could be continued to some (non-maximal) standard solution y, then we could apply the above argument to the initial value $\langle a, y(a)\rangle$.

This concludes the proof of Theorem 6.2 for standard F. By Transfer, the theorem is true for all F. □

7 Final Remarks.

Remark 7.1 The proofs of the global Peano theorem we found in the literature often simply appeal to Zorn's lemma (eg Ganesh [3], Theorem 4.7). The more careful proofs depend on **ADC**, usually without mentioning it explicitly. Hale [4] in his proof of global Peano theorem (Theorem 2.1, page 17) writes:

> "... there is a monotone increasing sequence $\{b_n\}$ constructed as above so that the solution $x(t)$ of (1.1) on $[a,b]$ has an extension to the interval $[a,b_n]$ and $(b_n, x(b_n))$ is not in $\bar{V}_n$. Since the b_n are bounded above, let $\omega=\lim_{n\to\infty} b_n$. It is clear that x has been extended to the interval $[a,\omega)$ ..."

What is actually clear is that his construction yields solutions $x_n(t)$ on $[a,b_n]$ for each n, and each $x_n(t)$ has extensions to some $x_{n+1}(t)$. The axiom **ADC** is needed to justify the existence of $x(t)$. Similarly Hartman [6, II, 3.1, page 13] constructs an increasing sequence $\{b_n\}$ such that any solution on $[a,b_n]$ has an extension to a solution on $[a,b_{n+1}]$. Here **ADC** is needed to justify the existence of a solution on $[a,\omega_+]$ for $\omega_+=\lim_{n\to\infty} b_n$. In Hartman's proof of III, Lemma 2.1, a key step to the proof of III, Theorem 2.1 (Osgood's theorem), **ACC** is used implicitly to obtain the sequence $\{u_n(t)\}$. Similar unacknowledged use of **ADC** appears in Kurzweil [14, pages 355–356].

Remark 7.2 Simpson [17] carried out a thorough study of the axioms needed to prove the *local* versions of Peano and Osgood theorems. He showed that (over $\mathbf{RCA}_0$) the local Peano theorem is equivalent to $\mathbf{WKL}_0$ and the local Osgood theorem is equivalent to $\mathbf{ACA}_0$ (see Simpson [18] for the description of these systems of second order arithmetic and additional information). In particular, the proofs of local versions of these theorems do not need any form of **AC**.

Remark 7.3 The conservativity of **SPOT** over **ZF** and the results of this paper imply that global Peano and Osgood theorems are provable in **ZF**.

In a discussion on MathOverflow [5], James Hanson pointed out that the same conclusion follows from Shoenfield's absoluteness theorem. A consequence of this theorem is that every Π^1_4 sentence provable in **ZFC** is provable in **ZF** alone. The global Peano theorem can be expressed by a Π^1_4 sentence, and therefore it is provable in **ZF**. The **ZF** proof obtained by conversion of the **ZFC** proof by this method is far from elementary; in addition to Shoenfield's absoluteness theorem, it relies on the notion of relatively constructible sets.

Clarification of a point in [10]. In Section 4 of [10], $\mathfrak{M}$-*generic filters* on a forcing notion $\mathbb{P} \in \mathfrak{M}$ are defined (see Definition 4.10). Following a paragraph that explains how such filters are constructed, it is stated that "$\mathfrak{M}$–generic filters $\mathcal{G} \subseteq M \times M$ on $\mathbb{H}$ are defined and constructed analogously." There is a difference though, in that the forcing notion $\mathbb{H}$ is a proper class from the point of view of $\mathfrak{M}$. The $\mathfrak{M}$–generic filters $\mathcal{G} \subseteq M \times M$ on $\mathbb{H}$ have to meet every class $D \subseteq M$ which is definable in $\mathfrak{M}$ (with parameters from M) and dense in $\mathbb{H}$. As there are only countably many such classes, the construction of a generic filter on $\mathbb{H}$ can proceed analogously to the construction of a generic filter on $\mathbb{P}$.

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