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Kuratowski's problem in constructive Topology[1]

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Abstract: A classical result by Kuratowski states that there are at most seven different combinations of the operators of interior and closure in a topological space, which become fourteen if one consider also complement. Two (and hence, usually, more) of these operators can coincide in some special classes of spaces; for instance, Boolean spaces have only six different combinations. This is the classical picture. What happens to this picture if it is looked at from a constructive point of view? The present paper provides an answer to this question, while leaving some problems open. The first part of the paper provides a constructive account of the closure–interior problem and discusses some special classes of spaces. The role of the set-theoretic (pseudo)complement is considered in the second part. The paper ends by showing what the Kuratowski's problem looks like in a pointfree framework, that is, within the theory of locales. This last part of the paper is independent from the underlying metatheory, although it is obtained by applying the constructive results in the previous parts.

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1 Introduction (and the classical Kuratowski's problem)

In classical topology, Kuratowski's closure–complement result says that there are at most 14 distinct combinations of the operators of closure c and complement $-$ on the subsets of a topological space; in this context, the operator of interior i appears as the composition $-c-$. Such fourteen operators form an ordered monoid (with respect to

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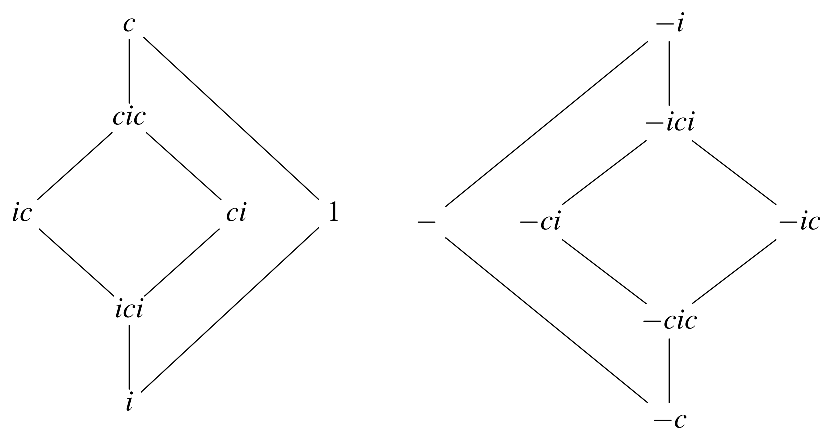
composition and pointwise ordering), whose Hasse diagram is the following:

(1)



(Here 1 is the identity operator).

The proof of this well-known result will follow from the constructive results below (Proposition 3.1) and from the identities $i- = -c$ and $c- = -i$ that hold when classical logic is assumed. Note that there are 7 distinct operators involving c and i only (that is, with an even number of occurrences of $-$); the other 7 are obtained by complement. The number of operators cannot be reduced, in general. For instance, the subset $\{-2\} \cup (-1,0) \cup (0,1) \cup \big((2,3) \cap \mathbb{Q}\big)$ of the real line $\mathbb{R}$ (with its usual topology) has 14 different images along the 14 operators.

A topological space where two or more of the operators in (1) collapse is typically "almost" discrete. (See [6, Theorem 2.1] for a precise statement; see also Section 2.7 below.)

The main aim of this paper is to find and discuss a constructive version of these results.

Here by "constructive" we simply mean that we use neither the Axiom of Choice nor the full Law of Excluded Middle (LEM), unless explicitly indicated, so that all our arguments will be intuitionistically (and topos-theoretically) valid.

One of the main consequences of adopting such a foundational standpoint is that the interior operator i cannot be expected to be defined in terms of closure and complement as $-c-$,[3] and hence it will appear explicitly in our treatment. Moreover, we have to deal with at least two non equivalent definitions of closure in a topological space: one "negative", in terms of complement and interior, and the other "positive", in terms of adherent points.

[3] In a discrete space this would imply $-- = 1$, that is, LEM.

The following section provides a constructive account of the closure–interior problem and some related results (about almost discrete spaces, for instance); here the closure operator is understood in its positive form, and the set-theoretic (pseudo)complement plays no role.

The pseudocomplement will appear in Section 3, where the negative closure operator is considered. There the interior–complement problem is studied, as opposed to the closure–complement problem of the classical approach.

In the last section we will study what the Kuratowski's problem looks like in a pointfree framework, that is, within the theory of locales. Mathematically, such a problem is quite different from that on topological spaces. The reason, roughly speaking, is that there are "more" (in a precise sense) locales than (sober) topological spaces; in particular, the lattice of all sublocales of any given space is richer than the lattice of its subspaces (because there exist non-trivial sublocales with no points). The sublocales of a given locale form a co-frame, rather than a frame; so a co-pesudocomplement has to be considered.[4] The fact that not all sublocales are complemented makes the pointfree version of the Kuratowski's problem closer, although in a dual sense, to the constructive than the classical problem for spaces. This is why the main result of the last section, although independent from the underlying logic, is derived from the constructive results in the previous ones.

Our work has been deeply influenced by the approach to constructive mathematics that Giovanni Sambin has been proposing for the past three or four decades and that will be extensively described in the oncoming book [10]: part of the material and ideas in this paper is due to him and will appear therein.

2 The closure–interior problem

In this section we study the closure–interior problem, that is, we consider all possible compositions of the two operators of closure c and interior i, and we order them pointwisely. In this way we obtain an ordered monoid (generated by c and i), which

[4] A frame [9] is a complete lattice in which binary meets distribute over arbitrary joins; every frame is an Heyting algebra and hence it has a pseudocomplement. A co-frame is a complete lattice in which binary joins distribute over arbitrary meets; in other words, it becomes a frame when its order is reversed. Here we are considering the natural order on sublocales, that is, their category-theoretic order as subobjects; such an order is the opposite of the pointwise order on nuclei (as functions).

we could call the general Kuratowski monoid.[5] For the sake of generality (and also in view of the application in the last section) we consider here operators on an arbitrary poset rather than simply on the powerset of a topological space.

2.1 The closure–interior problem on a poset

Given a poset $(L, \leq)$, a closure operator on L is an idempotent, monotone (that is, non-decreasing) function $c\colon L \to L$ such that $1 \leq c$. So a function $c\colon L \to L$ is a closure operator if and only if

(2) $$cx \leq cy \Leftrightarrow x \leq cy$$

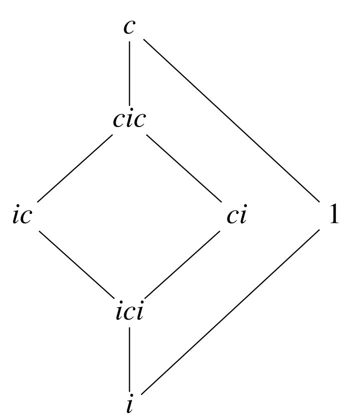
for all $x, y \in L$. Dually, an interior operator on L is an idempotent, monotone function $i\colon L \to L$ such that $i \leq 1$; and $i\colon L \to L$ is an interior operator if and only if

(3) $$ix \leq iy \Leftrightarrow ix \leq y$$

for all $x, y \in L$.

It turns out that the left-hand side of picture (1), namely (4) below, applies also in this general framework.

(4)



Proposition 2.1 *Let $(L, \leq)$ be a poset, c a closure operator on L, and i an interior operator on L. Then there are 7 different combinations of c and i, which are related as shown in (4) with respect to pointwise ordering. No other relation holds in general.*

[5] In [6] the term 'Kuratowski monoid' refers to the ordered monoid generated by c, i and $-$, possibly subject to some extra condition. Classically this makes no big difference: adding $-$ amounts in adding a flipped copy of the ordered monoid generated by c and i alone. Intuitionistically, of course, the pseudocomplement $-$ greatly increases the complexity of the generated monoid, as we will see in Section 3.

Proof (The statement follows from [5, Lemma 2.6] and a sketchy proof is given here only for the sake of self-containedness.) Since c and i are idempotent, there are only two combinations of c and i of length 2, namely ic and ci; clearly $i \leq ic, ci \leq c$. Similarly, there are two combinations of length 3, namely ici and cic, and $ici \leq ic, ci \leq cic$. Moreover, $i \leq ici$ since it is equivalent to $i \leq ci$ (by 3); and $cic \leq c$ by a dual argument.[6] Finally, any combination of length $n \geq 4$ is equivalent to a shorter one; the reason is that ic and ci are idempotent. Indeed $ic \leq icic$ because it is equivalent to $ic \leq cic$ (by 3); and $icic \leq ic$ follows from $cic \leq c$ as i is monotone. In a dual way, it is $ci = cici$.

Showing that no further inequality holds (in the most general case of a poset with operators) is pretty tautological. Let L be a formal copy of the 7-element poset (4); for convenience, we write the 7 different elements of L as a_x with $x = i, ici, ic, ci, 1, cic, c$ in the obvious way. Let i and c be the operators on L defined as $i(a_x) = a_{ix}$ and $c(a_x) = a_{cx}$. Then it is straightforward to check that i and c are an interior and a closure operator on L, respectively, and that the images of a_1 along i, ici, ic, ci, cic, c and 1 are related precisely as prescribed by (4) (trivially); in particular, they are all different. □

A more deep way to show that the 7 combinations in (4) are different, and that remain different even if further topological assumptions are made (such as those that we will add in the sequel), is to look for a topological interpretation, of course: the subset $\{-2\} \cup (-1,0) \cup (0,1) \cup \big((2,3) \cap \mathbb{Q}\big)$ of the real line has 7 different images along the 7 operators (where i and c are interpreted as the usual topological interior and closure in $\mathbb{R}$).

Perhaps surprisingly, Proposition 2.1 depends neither on LEM nor on topological notions as strictly understood (for instance, c need not preserve finite joins); moreover, no link at all is required between c and i (besides the fact that they operate on the same poset, of course).

From an algebraic point of view, (4) is the Hasse diagram of the idempotent, ordered monoid on two generators i and c subject to the relations $i \leq 1 \leq c$.

In Section 3 we will see what can be said constructively when we introduce the set-theoretic pseudocomplement. In what follows, instead, we shall investigate some classes of spaces in which some of the operators in (4) collapse.

[6] Clearly, there is a duality at work here. Indeed, every interior (closure) operator on a poset $(L, \leq)$ is a closure (interior) operator on its opposite $(L, \geq)$.

2.2 Kuratowski monoids

All possible Kuratowski monoids are obtained from (4) by imposing some extra condition on the 7 operators (in the form of some inequality between them). So, in order to understand the shapes of such quotient monoids, one needs to understand the implications among the several inequalities.

If i and c are the interior and closure operators on a topological space, and if classical logic is assumed, then there are only 6 possible Kuratowski monoids [6, Theorem 2.1]. Here we try to address the general case of a poset. So, in particular, we assume no link between i and c. In Section 2.3, on the contrary, we shall add some link that holds true in constructive topology.

By inspecting diagram (4), one sees that there are, a priori, 26 additional inequalities which could be imposed, namely:

$$\begin{array}{cccccc}
c = i & c = ici & c = ic & c = ci & c = cic & c = 1 \\
cic = i & cic = ici & cic = ic & cic = ci & cic \leq 1 & 1 \leq cic \\
ic = i & ic = ici & ic \leq ci & ic \leq 1 & 1 \leq ic & \\
ci = i & ci = ici & ci \leq ic & ci \leq 1 & 1 \leq ci & \\
ici = i & ici \leq 1 & 1 \leq ici & & & \\
1 = i & & & & &
\end{array}$$

In the above list, we have preferred using equations in all those cases in which one of the inequalities holds by (4). Actually, all inequalities above can be replaced by equations, thanks to the following lemma.

Lemma 2.2 *Let $(L, \leq)$ be a poset, c a closure operator on L, and i an interior operator on L. Then the conditions listed in the same row of Table (5) are equivalent to each other.*

(5)
$$
\begin{array}{rcc}
cic \leq 1 & cic = i & \\
1 \leq cic & c = cic & \\
ic \leq ci & cic = ci & ic = ici \\
ic \leq 1 & ic = i & \\
1 \leq ic & c = ic & \\
ci \leq ic & cic = ic & ci = ici \\
ci \leq 1 & ci = i & \\
1 \leq ci & c = ci & \\
ici \leq 1 & ici = i & \\
1 \leq ici & c = ici &
\end{array}
$$

Proof In the whole proof we make free use of the facts we already know from Proposition 2.1 and its proof.

The following equivalences follow directly from the fact that i is an interior operator and c is a closure operator, that is, from (3) and (2).

- $1 \leq cic$ if and only if $c \leq cic$ if and only if $c = cic$
- $ic = ici$ if and only if $ic \leq ici$ if and only if $ic \leq ci$ if and only if $cic \leq ci$ if and only if $cic = ci$
- $ic \leq 1$ if and only if $ic \leq i$ if and only if $ic = i$
- $1 \leq ci$ if and only if $c \leq ci$ if and only if $c = ci$
- $ici \leq 1$ if and only if $ici \leq i$ if and only if $ici = i$

The inequality $cic \leq 1$ implies both $ic = icic \leq i$ and $ci = cici \leq i$, that is, $ic = i = ci$; therefore $cic \leq 1$ implies $cic = i$, and vice versa. Similarly, $1 \leq ici$ implies both $c \leq ic$ and $c \leq ci$, that is, $ic = c = ci$; so it is equivalent to $c = ici$.

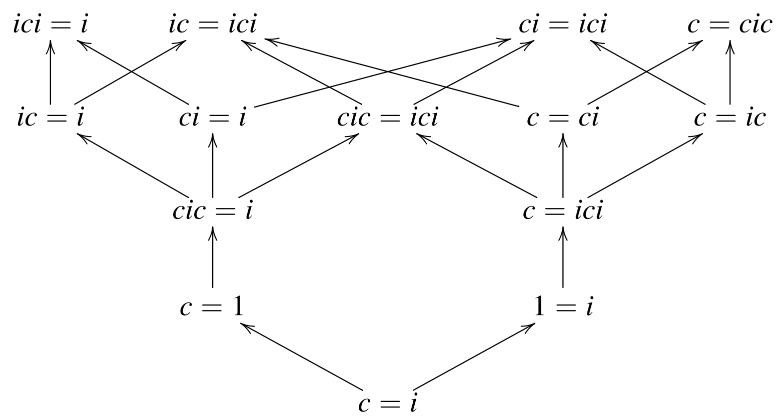
The inequality $1 \leq ic$ implies $c \leq icc = ic$, that is, $c = ic$ which implies back $1 \leq ic$. Similarly, $ci \leq 1$ implies $ci \leq i$, that is, $ci = i$ which implies back $ci \leq 1$.

Finally, $ci \leq ic$ implies $cic \leq ic$, that is, $cic = ic$ which implies back $ci \leq ic$. Also, $ci \leq ic$ implies $ci \leq ici$, that is, $ci = ici$ which implies back $ci \leq ic$. □

Thus we can delete the 10 inequalities from our 26–item list. Moreover, we now know that $cic = ci$ is equivalent to $ic = ici$, and $cic = ic$ is equivalent to $ci = ici$. So we are left with the following 14 equations: $c = i$, $c = ici$, $c = ic$, $c = ci$, $c = cic$, $c = 1$, $cic = i$, $cic = ici$, $ic = i$, $ic = ici$, $ci = i$, $ci = ici$, $ici = i$, and $1 = i$.[7]

Proposition 2.3 *Let $(L, \leq)$ be a poset, c a closure operator on L, and i an interior operator on L. Then the implications which hold in general between the 14 conditions above are shown in (6).*

(6)



Proof Some of the implications in (6) are direct consequences of the ordering shown in (4); for instance, $c = i$ clearly implies all others, since in that case the poset (4) collapse to a one-point poset. Some others are straightforward; for instance, if $i = 1$, then all combinations with at least one occurrence of c become equivalent to c. Less trivial implications are proven as follows. From $cic = i$ one gets $ic = icic = ii = i$ and $ci = cici = ii = i$. Similarly, $ici = c$ implies both $ci = c$ and $ic = c$. Therefore both $cic = i$ and $ici = c$ yield $cic = ici$. Finally $cic = ici$ implies $ic = icic = iici = ici$ and $ci = cici = icii = ici$. □

One could also be interested in equations which are not explicitly shown in (6) as they arose by putting together two (or more) of those shown in the diagram. However, every such compound equation is already present, up to equivalence, in the diagram. A couple of notable examples follow.

- $c = i$ if and only if $cic = 1$ if and only if $ic = 1$ if and only if $ci = 1$ if and only if $ici = 1$

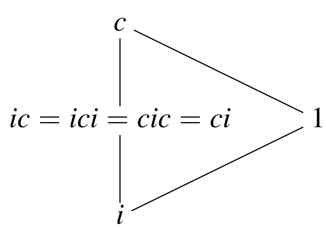
[7] We will not claim that this number cannot be reduced further, although we have some reason (see Section 2.6) to think it cannot.

Every equation of the form $cx = 1$ implies $c = 1$ (because $c = c1 = ccx = cx = 1$), and hence also $x = 1$. Similarly, every equation of the form $ix = 1$ implies $i = 1 = x$. So $ic = 1$ yields $i = 1 = c$ and hence it is equivalent to $c = i$. Similarly, $ci = 1$ is equivalent to $c = i$ too. Also, $ici = 1$ yields $i = 1 = ci$ and hence $i = 1 = c$; so, again, it is equivalent to $c = i$. And similarly for $cic = 1$.

- $cic = ici$ if and only if $ic = ci$

The equation $cic = ici$ yields $ic = ici = ci$ by (6); conversely, if $ic = ci$, then $cic = cci = ci = cii = ici$.

For instance, the classical Sierpiński space satisfies $ic = ci$, and hence its Kuratowski monoid has the following form.



This provides also some counterexamples: it shows that $ic = ici = cic = ci$ implies neither $ic = i$ nor $ci = i$ nor $ci = c$ nor $ic = c$.

2.3 Approaching the topological case: posets with overlap

In this section we gradually approach the topological case, but without actually reaching it, that is, we shall still work in a pretty general order-theoretic framework. So, we gradually require additional properties (and structure) on the poset L and the operators i and c: topological spaces as constructively understood will always be the motivating examples.

From now on let us suppose that L is equipped with an **overlap relation** according to the following definition.[8]

Definition 2.4 A *poset with overlap* is a poset $(L, \leq)$ equipped with a binary relation $\between$ such that:

[8] As far as we know, the recognition of the importance of the overlap relation in constructive mathematics, its axiomatization, and its very notation are due to Giovanni Sambin. For more about posets, lattices, and frames equipped with an overlap relation we refer the reader to [5, 4, 3, 2, 10] and to the other works on overlap algebras cited therein.

- if $x \between y$, then $y \between x$ (*symmetry*)
- if $x \between y$ and $y \leq z$, then $x \between z$ (*monotonicity*)
- if, for all $z \in L$, $z \between x$ implies $z \between y$, then $x \leq y$ (*density*)

for all $x, y, z \in L$.

When L is a powerset, $x \between y$ means that $x \cap y$ is inhabited (classically, $x \cap y \neq \emptyset$), that is, x and y overlap.

When we consider interior i and closure c on a poset with overlap, we always assume that i and c are linked by the following condition:

- if $ix \between cy$, then $ix \between y$ (*compatibility*)

for all $x, y \in L$. (So, for the first time, we require a link between i and c; and, being such a link asymmetric, we break the duality that has applied till now.)

In a topological space, compatibility follows from the usual (constructive) definition of c in terms of adherent points. Indeed, if a point belongs to the open set ix, so that ix is one of its open neighbourhoods, and, at the same time, it belongs to the closure cy of y, then y and the open neighbourhood ix has to overlap each other by the very definition of c.

Lemma 2.5 *Let L be a poset with overlap, and let i and c be interior and closure operators on L. If i and c are linked by compatibility, then $c = cic$ implies $ici = i$.*

Proof Assume $c = cic$. We have to check that $icix \leq ix$ for every $x \in L$ (the converse inequality being trivial). By density, it is sufficient to check that $z \between icix$ implies $z \between ix$ for all $z \in L$. So let $z \between icix$. By (symmetry and) monotonicity, we have $cz \between icix$, and hence $cicz \between icix$ by assumption. So $icz \between icix$ by (symmetry and) compatibility and hence $icz \between cix$ by monotonicity. By applying compatibility again we get $icz \between ix$ and hence $cz \between ix$ by (symmetry and) monotonicity. A last application of (symmetry and) compatibility gives $z \between ix$ as wished. □

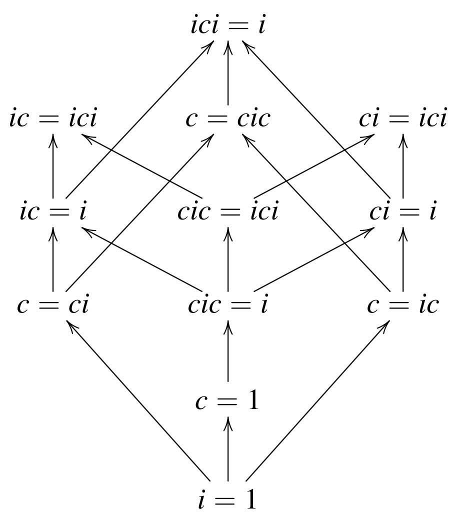
Proposition 2.6 *Let L be a poset with overlap, and let i and c be interior and closure operators on L linked by compatibility. Then the following hold:*

- *if $c = ic$, then $ci = i$*
- *if $c = ci$, then $ic = i$*
- *if $c = ici$, then $c = i$*

Proof All items follow easily from the implications in (6) together with the previous lemma. For instance, if $c = ic$, then $c = cic$ and hence $ici = i$; so $ci = (ic)i = i$. □

Under the present assumptions, therefore, (6) can be simplified and redrawn as follows

(7)



where $c = ici$ and $c = i$ are now equivalent to $i = 1$.

2.4 Closure and interior on a semilattice with overlap

Let L be a poset with overlap. We now assume that L has all finite meets; we write $x \wedge y$ for the meet of x and y, and $\top$ for the top element. Also, we assume a further condition on the overlap relation, namely

$$x \between y \text{ if and only if } (x \wedge y) \between \top$$

for all $x, y \in L$. In this case, we call L a *semilattice with overlap*.

Moreover, we require that i fixes $\top$,

$$i\top = \top$$

and we show that, in this case, we can contract diagram (7) further.[9]

Proposition 2.7 *Let L be a semilattice with overlap, i an interior operator on L such that $i\top = \top$, and c a closure operator on L compatible with i. Then, in addition to the properties in Proposition 2.6, we also have:*

- *$c = cic$ implies (and so it is equivalent to) $c = ic$;*

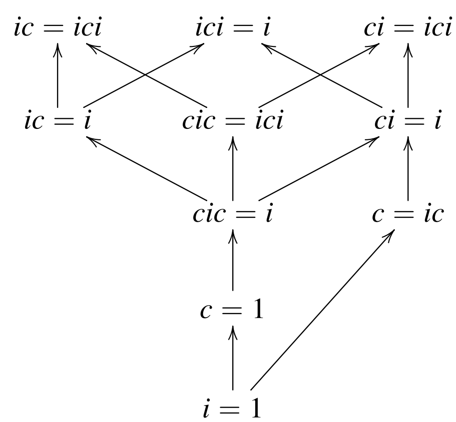
[9] Requiring $i(x \wedge y) = ix \wedge iy$ would be quite natural too, although it does not seem to lead to further simplification.

- *$c = ci$ implies (and so it is equivalent to) $i = 1$.*

Proof The second item follows from the first; indeed, $c = ci$ implies both $ic = i$ and $c = cic$; so we have $c = ic = i$ by the first item, and hence $i = 1$. The first item can be proven as follows. First, by (5), we rewrite the premise as $1 \le cic$ and the conclusion as $1 \le ic$. Then we check that $x \le icx$ by means of density. Assume $z \between x$, that is, $(z \wedge x) \between \top$. By our premise, this gives $cic(z \wedge x) \between \top$. Since $\top = i\top$, we have $ic(z \wedge x) \between \top$ by compatibility, and hence $(cz \wedge icx) \between \top$ by monotonicity.[10] This means $cz \between icx$ and, by compatibility again, we get $z \between icx$ and we can conclude. □

Under the present assumptions, therefore, (7) becomes (8).

(8)



Classical Kuratowski monoids. We pause here and derive the classical result for topological spaces. In that case, in fact, we have $-i = c-$ and $-c = i-$ so that $c = -i-$ and $i = -c-$; here $-$ can be modelled as an involutive, antitone map on L. Under such assumptions, one has the following implications:

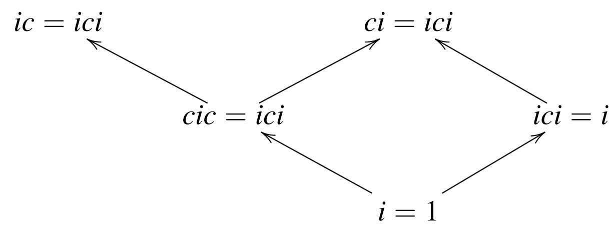
- if $ici = i$, then $c = ic$;
- if $ic = i$, then $i = 1$.

Indeed, these are just the two items in Proposition 2.7, although written in a classically equivalent form ($ici = i$ is equivalent to $-ici- = -i-$, that is, $cic = c$; and $ic = i$ is equivalent to $-ic- = -i-$, that is, $ci = c$).

[10] Clearly, $ic(z \wedge x) \le icz \le cz$ and $ic(z \wedge x) \le icx$.

Therefore, under the present assumptions (so, in particular, in every topological space within a classical metatheory), diagram (8) reduces to

(9)



and no other implications hold in general. For instance, the Sierpiński space satisfies $ci = ic$, that is, $cic = ici$ while $ici \neq i$.

Each of the 5 conditions shown in the diagram (9) corresponds to a particular class of spaces and gives rise to a corresponding Kuratowski monoid. For instance, the monoid corresponding to a Boolean space ($ici = i$) has just three elements, namely $i \leq 1 \leq c$, because $c = cic = ic$ and $ci = ici = i$. We refer the reader to [6] for further details.

2.5 Kuratowski's problem for Kolmogorov spaces

In this section, in order to simplify diagram (8), we restrict our attention to T_0 (Kolmogorov) spaces. Throughout this section, therefore, L will be the powerset of X, with X a topological space, and i and c will be the topological interior and closure operators on it. Here X is said to be a Kolmogorov space if

$$c\{x\} = c\{y\} \to x = y \tag{T_0}$$

for all $x, y \in X$.

We are going to show that in this case $c = ic$ becomes equivalent to $i = 1$.[11] Actually, we can prove a bit more. First, let us consider the following separation properties for a space X:

- $x \in c\{y\} \to y \in c\{x\}$, for all $x, y \in X$ $\quad (R_0)$
- $c\{x\} = x$, for all $x \in X$ $\quad (T_1)$

Clearly T_1 is the conjunction of T_0 and R_0. Then we have the following (see [2]):

(1) for any topological space, $ci = i$ implies R_0

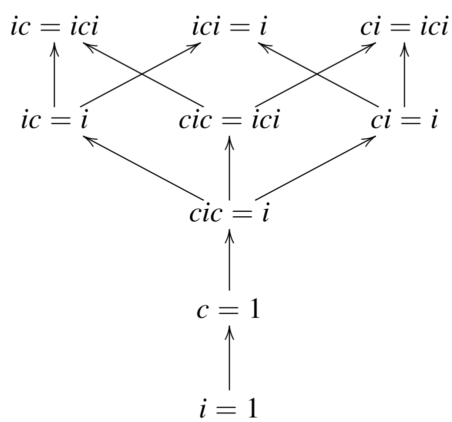
[11] Classically, the restriction to Kolmogorov spaces is perhaps undesirable as it makes even the condition $ici = i$ (Booleanness) boil down to $i = 1$ (discreteness).

(2) for any Kolmogorov space, $ci = i$ implies T_1[12]
(3) for any Kolmogorov space, $c = ic$ implies $i = 1$

Indeed, item (3) follows from item (2) because $c = ic$ implies $ci = i$ and hence, by T_1, we have $i\{x\} = i(c\{x\}) = c\{x\} = \{x\}$. Item (2) follows from item (1) since R_0 is equivalent to T_1 in the presence of T_0. Lastly, item (1) can be proven as follows. Assume $x \in c\{y\}$ and let A be any open neighbourhood of y; then $x \in c\{y\} \subseteq cA = A$ (because $ci = i$). Thus every open neighbourhood of y overlaps $\{x\}$, that is, $y \in c\{x\}$.

In the case of a Kolmogorov space, therefore, diagram (8) becomes

(10)



because $c = ic$ is now equivalent to $i = 1$.

2.6 A Brouwerian counterexample

In [2], a family of topologies $(2, \tau_p)$ on the set $2 = \{0, 1\}$ is constructed, with p ranging over the collection of intuitionistic truth values Ω. There it is shown that $c = 1$ holds in every such space, so that they are T_1 (and even T_2). Moreover, $(2, \tau_p)$ is discrete if and only if $p \vee \neg p$ holds. So assuming that $c = 1$ implies $i = 1$ for all topological spaces is equivalent to LEM:

$$(c = 1 \to i = 1) \iff LEM$$

[12] Although it is not strictly related to the thread of our discussion, it is perhaps worth noting that $ci = i$, for a T_0 space, implies even that the space is Hausdorff (T_2), provided that T_2 is defined as follows: if A and B overlap for all opens $A \ni x$ and $B \ni y$, then $x = y$ (which is classically equivalent to the usual formulation of T_2, of course). Indeed, the premise of T_2 means that $x \in c(B)$ for all open $B \ni y$; under the assumption $ci = i$, this becomes $x \in B$ for all open $B \ni y$, that is, $y \in c\{x\}$; therefore $y = x$ because T_1 follows from $ci = i$.

We refer the interested reader to [2] for the construction of $(2, \tau_p)$ and the proof of its properties. Here we just recall the definition of τ_p for the readers who want to try to work out the details themselves: put $P = \{x \in 2 \mid p\}$ and consider the topology τ_p whose basic (sic) open sets are $\{0\} \cap P$, $\{1\} \cap P$, $\{0\} \cup P$, and $\{1\} \cup P$.

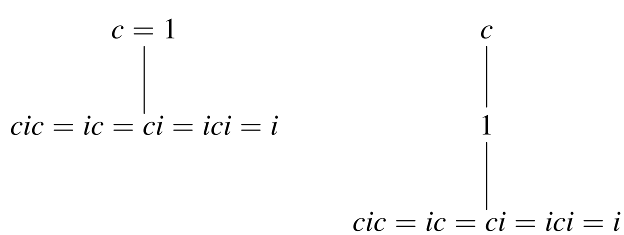
As a consequence, none of the 8 conditions different from $i = 1$ in (10) can imply $i = 1$ constructively (some, of course, cannot imply it either classically).

2.7 On discrete and almost discrete spaces.

A consequence of the discussion in the previous sections of this chapter is that there are a number of conditions on c and i which force a Kolmogorov space to be discrete and so they are all equivalent to $i = 1$. For convenience, we collect them all here: $i = 1$, $c = i$, $ici = 1$, $ic = 1$, $ci = 1$, $cic = 1$, $1 \leq ici$, $1 \leq ic$, $1 \leq ci$, $1 \leq cic$, $c = cic$, $c = ic$, $c = ci$, $c = ici$. On the contrary, as we know from the counterexample above, there are other conditions that cannot imply $i = 1$ constructively, although they hold in all discrete spaces, an example being $c = 1$.

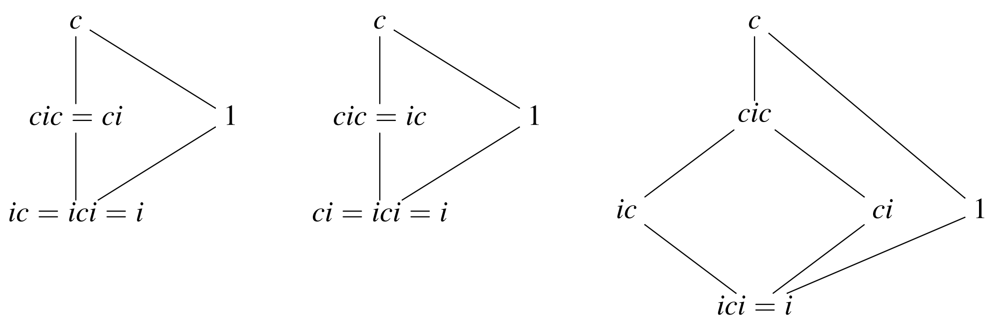
We now restrict our attention to the condition $ici = i$. Classically, a space that satisfies $ici = i$ is Boolean (see page 22); that is, its open sets form a complete Boolean algebra with respect to the usual set-theoretic operations.[13] And a Boolean, Kolmogorov space is just a discrete space [2]; hence a Boolean space can be considered almost discrete. Constructively, the picture is not so trivial. . .

The open sets of a space such that $ici = i$ form a (spatial) overlap algebra [2], a constructive variation on the notion of a Boolean locale.[14] So it makes sense to study Kolmogorov spaces where $ici = i$. Constructively, there are (at most) 6 possible Kuratowski monoids for such a space. One is the discrete case ($i = 1$) in which all operators coincide: this is the only possibility under classical logic. The other 5 are shown below.



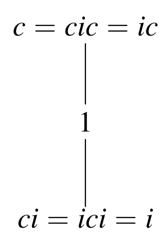
[13] Boolean spaces are also known as partition spaces [6].

[14] Our personal motivation in studying the condition $ici = i$ lies precisely in the fact that it is connected to our work on overlap algebras [5, 4, 3, 2]. In view of the theory of overlap algebras, one could claim that $ici = i$ is a natural constructive counterpart to Booleanness.



In view of the counterexample recalled above, we know that the first of the 5 monoids shown above cannot collapse to the discrete case.

We end this section with a brief look at another condition, namely $c = ic$. Classically this is just another equivalent way to define a Boolean space. One feature that makes this condition interesting from a constructive point of view is that a Kolmogorov space satisfying it has to be discrete (like in the classical case), as shown in Section 2.5 above. Constructively, therefore, there can be only two Kuratowski monoids satisfying $c = ic$ (without assuming T_0 of course): one is shown below,



and the other is the one-point monoid corresponding to the discrete case.

3 The interior-(pseudo)complement problem

Aim of this section is to address the issue of dealing with the pseudocomplement. Again, we prefer working in an abstract algebraic setting.[15] Accordingly, throughout this section we assume our poset L to be equipped with a pseudocomplement operator $-$, that is, a function $- : L \to L$ such that

$$x \leq -y \text{ if and only if } y \leq -x$$

[15]The difficult problem of studying all possible combinations of i, c and $-$ in an abstract setting, and under various assumptions, was tackled in [1]; there several cases are classified by means of a substantial use of computer-aided methods.

for all $x, y \in L$. Standard arguments show that the following properties hold in every poset with pseudocomplement:

- $x \leq --x$ $(1 \leq --)$
- if $x \leq y$, then $-y \leq -x$ $(-$ is antitone$)$
- $---x = -x$ $(--- = -)$
- $x \leq --y$ if and only if $--x \leq --y$ $(--$ is a closure operator$)$

(for $x, y \in L$). In particular, $-$ defines an antitone Galois connection on L.

As usual, let i be an interior operator on L. We put $b = -i-$.

It is easy to check that b is a closure operator.[16] So we can apply Proposition 2.1 to i and b and get (at most) 7 possible combinations of i and b (with no further occurrence of $-$) which are arranged as in diagram (4) (where c's have to be replaced by b's, of course).

The problem is now to understand what happens when further occurrences of $-$ are allowed. We claim that the possible combinations amount to 31, at most, as listed in table (11) below: 17 are monotone (and they include the 7 combination of i and b just mentioned), while 14 are antitone.

[16] In constructive topology, b is different from the closure operator c defined via adherent points; actually it is $c \leq b$, but not the other way around. Note also that $-b-$ cannot coincide with i constructively; in fact, it is not even an interior operator, as $-b- \leq 1$ does not hold in general (think of the case $i = 1$).

(11)

monotone
1 $--$
i $i--$ $--i$ $-b-$ $i--i$
b
ib $--ib$
bi $bi--$
ibi $ibi--$ $--ibi$ $--ibi--$
bib
17

antitone
$-$
$i-$ $-b$
$-i$ $b-$
$i-i$ $ib-$ $-bi$ $-bi--$
$-ib$
$ibi-$ $-bib$
$-ibi$ $-ibi--$
14

Some combinations can be written in different ways; here are two examples: $-ibi--$ $= -i-i-i--= bib-$; $-ib = -i-i- = bi-$. Classically it makes sense to require also $-- = 1$; combinations which coincide under such classical assumptions are put within the same box in the table above.

To show that there is no other combination which is different from those shown in the table, we can explicitly construct the Cayley graph of the monoid generated by i and $-$, provided that we understand what relations have to be imposed on the generators. Clearly, we know that $ii = i$ and $---- = --$, together with the other equations from Proposition 2.1 such as $ibib = ib$ and $bibi = bi$. Moreover, we can prove the following few facts, which will prove essential in reducing the number of possible combinations.

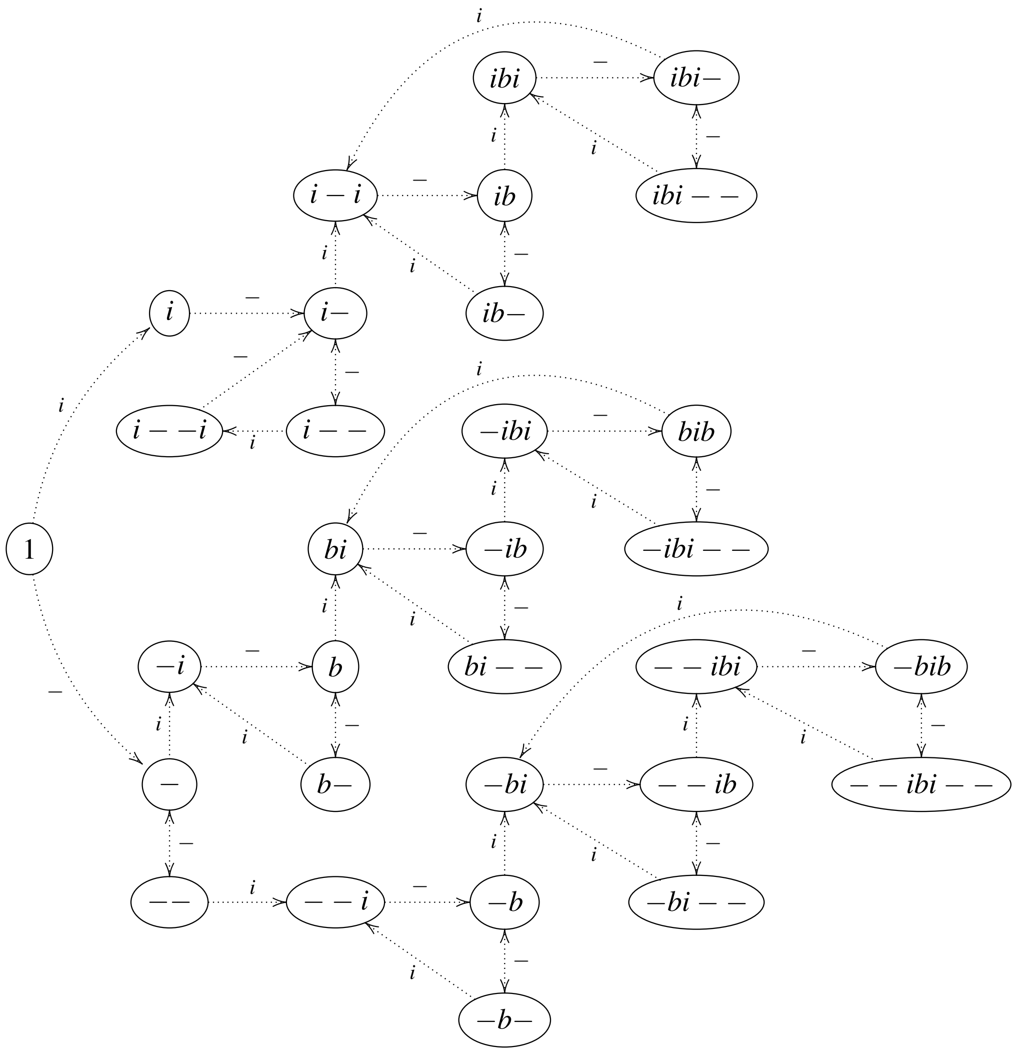
- $i-b = i--i- = i-$
  (This is a weak version of the classical equation $-b = i-$.) Since b is a closure operator (and $-$ is antitone), $i-b \leq i-$ is clear. As for the other direction, we have $i- \leq i-b$ if and only if $i- \leq -b$ if and only if $b \leq -i-$ which is trivial.
- $b-i = -i--i = -i$

(This is a weak version of $b- = -i$.) As above, $-i \leq b - i$ holds because $1 \leq b$; while $b - i \leq -i$ follows from $i \leq i - -i$, which is true because $1 \leq --$ and i is monotone.

- $ib - i = i - i = i - bi$
  This is a direct consequence of the previous two facts.
- $i - i - i - i = i - i$
  The composition $i-$ works as a pseudocomplement on the sub-poset of i-fixed elements[17] of L; indeed $ix \leq i - iy$ if and only if $ix \leq -iy$ if and only if $iy \leq -ix$ if and only if $iy \leq i - ix$. In particular, $(i-)^3 = (i-)$ when applied to i-fixed elements.
- $i - ibi = i - i = ibi - i$
  This is another way to write the previous fact.
- $iwi = i - i$ for every word w written in the alphabet $\{i, -\}$ and containing precisely three occurrences of $-$.
  This follows by putting together the previous facts.
- $iwi = i - i$ for every word w written in the alphabet $\{i, -\}$ and containing an odd number of occurrences of $-$.
  By repeatedly applying the previous fact.
- iwi is either i or $i - -i$ or ibi for every word w written in the alphabet $\{i, -\}$ and containing an even number of occurrences of $-$.
  By the previous fact.

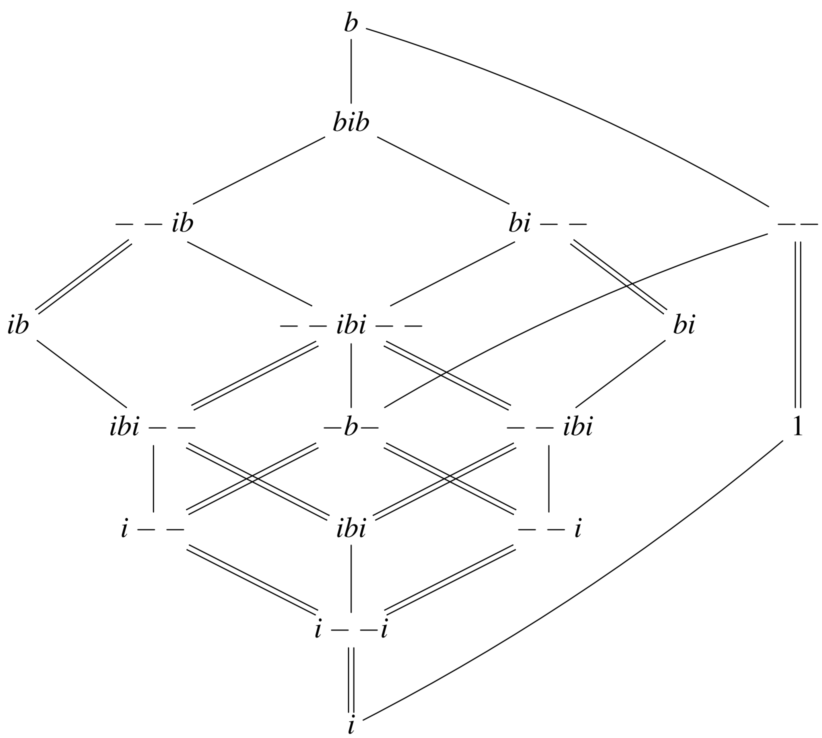
The following is our Cayley graph under the current assumptions. An arc with label l (with $l = i$ or $l = -$) connects the vertex w to the vertex wl; in this way, any w is the sequence of labels of the path from 1 to w. In order to increase readability, we have not drawn i-labelled loops on words of the form wi; moreover, we have written as a single left-right arrow what should be, in fact, a pair of opposite arcs, both labelled with $-$, between $w-$ $(= w - --)$ and $w - -$.

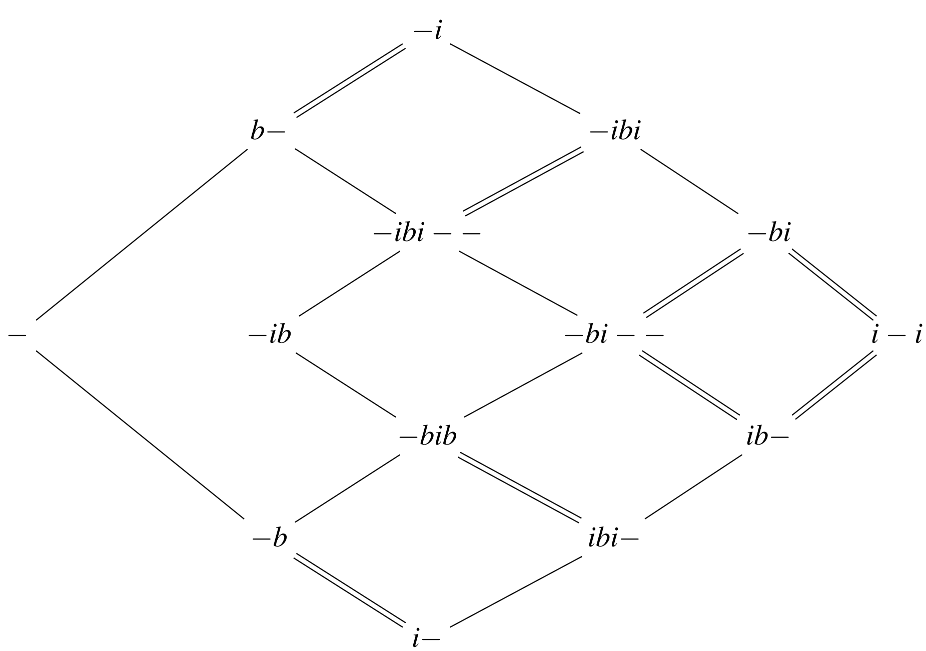
[17] Equivalently, this sub-poset is the image of the map i.



The corresponding 31–element ordered monoid is shown below (as expected, its Hasse diagram has two disconnected components): all inequalities are quite straightforward and we would give no detail. Summing up we have the following result.

Proposition 3.1 *Let $(L, \leq, -)$ be a poset with pseudocomplement and let i be an interior operator on L. Then there are at most 31 different combinations of i and $-$, which are related as shown below (where $b = -i-$).*





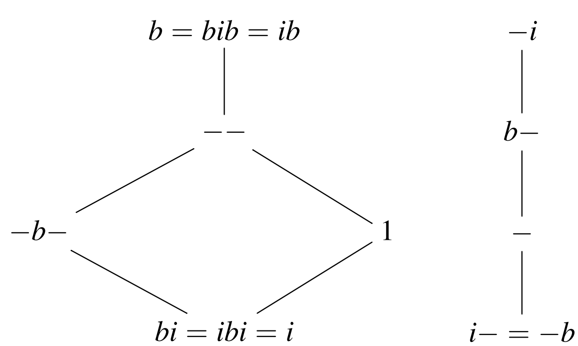
We have used a double line to connect two operators when they are equal classically, that is, under the extra assumption $-- = 1$. By collapsing such double lines one gets the classical picture (1).

Constructively, many of the inequalities between monotone operators (first half of the diagram) are seen to be strict because in the case $i = 1$ they would imply $-- = 1$.

Note that the first half of the diagram contains a copy of (4) for $c = --$, as expected since $--$ is a closure operator. We see no reason to expect some of these 7 operators to coincide. Why should there be any special link between i and $--$ in general? So, for instance, we do not expect $i-- \leq --i$ to hold in general; and hence we do not expect $-i \leq b-$ either. Similarly for other inequalities.

Some issue, however, is still open and further work is required to find proper Brouwerian counterexamples and check that all shown inequalities are strict, if that is the case.

Constructively, Boolean spaces are . . . trivial! In a Boolean space every open set is complemented in the lattice of opens, so that $ibi = i$ holds. However, since the lattice of opens is a sublattice of the lattice of subsets (joins and meets are computed in the same way, namely, as set-theoretic unions and intersections), every open is also complemented in the lattice of subsets; so $--i = i$. As a consequence, the two complements of an open set (as an open and as a subset) have to coincide, that is, $i-i = -i$.[18] In particular we have $-b = --i- = i-$, $ib = i-i- = -i- = b$ and $bi = -i-i = --i = i$. In a Boolean space, therefore, the number of combinations of i and $-$ reduces significantly; the corresponding monoid is shown below (only some notable equations appear explicitly).



[18] Here we are using the following simple fact, which holds in every Heyting algebra: an element x is complemented, that is, there exists y with $x \wedge y = \bot$ and $x \vee y = \top$ if and only if $x \vee -x = \top$ (that implies $--x = x$). So the complement of x, when it exists, is unique and coincides with its pseudocomplement $-x$.

However pleasant such a picture might look, it is pretty much useless! Indeed, in a sense, Boolean spaces are uninteresting from a constructive point of view (this is why other variants of Booleanness are considered in Section 2.7), because of the following fact [2, Proposition 2.1]: if there exists an inhabited Boolean space, then LEM holds.

To show this, assume (X, τ) is an inhabited Boolean space with $a \in X$. Let Ω be the set of truth values (the powerset of a singleton) and put:

$$F : \begin{array}{ccc} \tau & \longrightarrow & \Omega \\ A & \mapsto & a \in A \end{array} \qquad G : \begin{array}{ccc} \Omega & \longrightarrow & \tau \\ p & \mapsto & \{x \in X \mid p\} \end{array}$$

(The subset $B := \{x \in X \mid p\}$ is open; indeed, if $y \in B$, then p is true and hence $B = X$; so B is open and, in particular, y belongs to the interior of B.) Note that $F(X)$ is true (because $a \in X$) and that $F(A \cup B)$ is $F(A) \vee F(B)$. Moreover, $F(G(p))$ means $a \in \{x \in X \mid p\}$ and so it is equivalent to p. Similarly, $F(-G(p))$ is $\neg p$. Putting all these things together one gets $p \vee \neg p$ if and only if $F(G(p)) \vee F(-G(p))$ if and only if $F(G(p) \cup -G(p))$ if and only if $F(X)$ because τ is Boolean by assumption. So $p \vee \neg p$ is true.

In constructive topology, therefore, Boolean spaces are morally empty.[19] So, despite the fact that Boolean locales play a fundamental role in constructive pointfree topology (Isbell's density theorem [8]), Boolean spaces can only have a marginal role in constructive pointwise topology. For this reason, other variants of Booleanness (such as those mentioned in Section 2.7) are worthy of being studied.

4 The pointfree Kuratowski's problem

In this section, we look at the Kuratowski's problem from a pointfree perspective, that is, we consider sublocales of a given locale and the operators on them.[20] Accordingly, the role of our poset L is here played by the co-frame of sublocales (= the frame of nuclei) of a given locale/frame X.

From a foundational point of view, this section is quite independent from the underlying logic. Nevertheless, we here see an advantage of the constructive approach adopted

[19]To put it more formally, in a topos in which LEM fails (such as, for instance, the topos of sheaves over the Sierpiński space), the only Boolean space is the empty space.

[20]We refer the reader to [9, Chapter II] for a detailed introduction to locales. Apparently, the literature on the localic version of the Kuratowski's problem is quite poor. In [1] it is reported that the problem is studied in [7], but we were unable to retrieve that bibliographic source. Anyway, the result in this section agrees with what is reported in [1] of the work in [7], namely, that there are 21 combinations of the operators in the localic case.

above: the results obtained in the previous sections will come in handy here, since not all sublocales are complemented (so that the classical approach to the Kuratowski's problem does not apply).

Since sublocales form a co-frame, this time we have to deal with a co-pseudocomplement, that is, a map $-$ on L such that

- $-x \leq y$ if and only if $-y \leq x$

(that is, a pseudocomplement on the opposite order). It follows (see Section 3) that:

- $-- \leq 1$
- $-$ is antitone
- $---=-$
- $--$ is an interior operator on L

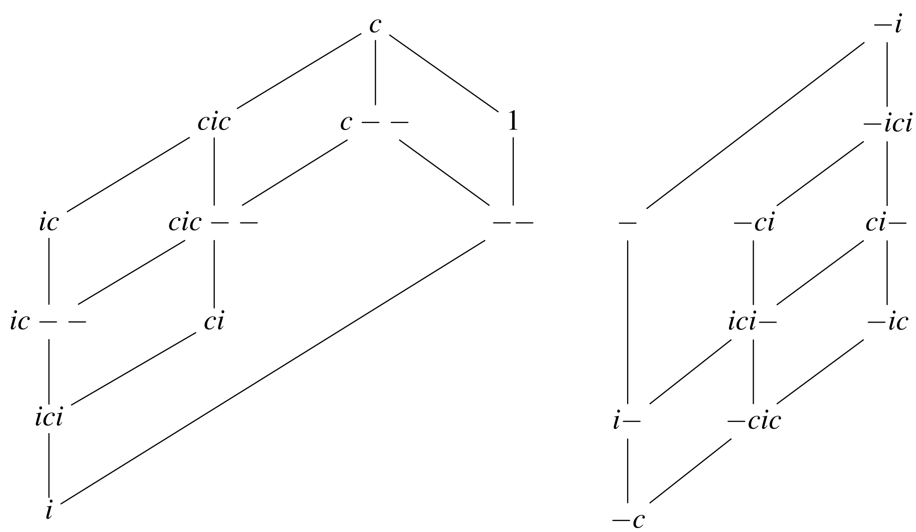
Recall from [9] that a nucleus j is just a closure operator that preserves binary meets in X; the (frame corresponding to the) sublocale defined by j is $X_j = \{x \in X \mid x = jx\}$. For instance, the identity map corresponds to X itself, and the constant map $x \mapsto \top$ (where $\top$ is the top element in X) corresponds to the smallest sublocale of X. An other important example is the double negation nucleus $x \mapsto \neg\neg x$ (where $\neg$ is the pseudocomplement in X, not to be confused with the co-pseudocomplement in L), which defines the smallest dense sublocale $X_{\neg\neg}$ of X.

Nuclei of the form $x \mapsto a \to x$, for $a \in X$, define open sublocales; nuclei of the form $x \mapsto x \vee a$ define closed sublocales. It is well known that open and closed sublocales are complemented in L: the open and closed sublocales defined by the same $a \in X$ are each other's complement. Given a sublocale X_j, its closure is the smallest closed sublocale containing it, and its interior is the greatest open sublocale contained in it. We write c and i for the operators that map each sublocale to its closure and interior, respectively; these are a closure operator and an interior operator on the co-frame of sublocales.

The facts about open and closed sublocales that we have just recalled imply $--i = i$ and $--c = c$ (because open and closed sublocales are complemented). Also we have $c-i = -i$ and $i-c = -c$ because the complement of an open sublocale is closed and the complement of a closed sublocale is open. As a consequence we get $c- \leq -i$ and $-c \leq i-$. So $c-- \leq -i- \leq c$ and $i = --i \leq -c- \leq i--$. Moreover, $i-- \leq i$ because $-- \leq 1$; so $i = -c- = i--$, hence $-i = c-$ and $-i- = c--$. All these properties are collected in the following two items.

- $--i = i = -c- = i-- \leq -- \leq c-- = -i- \leq c = --c$
- $-c \leq i- \leq - \leq c- = -i$

Formally, the Kuratowski's problem in this case is a special case of the interior-pseudocomplement problem studied in Section 3... provided that one is careful enough! Indeed, since now $-$ is a co-pseudocomplement, one has to apply Proposition 3.1 with respect to the opposite order, so that the roles of the interior and closure operators are switched. Indeed, a closure on $(L, \leq)$ is an interior on $(L, \geq)$, and an interior on $(L, \leq)$ is a closure on $(L, \geq)$; similarly, a co-pseudocomplement on $(L, \leq)$ is a pseudocomplement on $(L, \geq)$. As one of the above properties says that $i = -c-$, we fall under the assumption about b in Section 3 (where the closure b was supposed to be $-i-$). In view of this, we can take the diagrams in Section 3, reverse them (because the order is now reversed), write c in place of i, then (in this order!) write i in place of b, and apply all simplification implied by the above properties. Eventually, we get the following 21–element monoid.



A counterexample to $i- = -c$ and $-i- = c$. As before, we are not going to seek for all counterexamples to the above inequalities. However, we will make an exception: we check that $i- \not\leq -c$ and so $c \not\leq -i-$ (= $c--$), as opposite to the fact that both $-i = c-$ and $i = -c-$ hold. In what follows we can adopt a classical metatheory; in fact, we want to show $i-$ and $-c$ to be different even classically.

First, we need some extra detail about sublocales. Given a sublocale X_j, its closure is given by the nucleus $x \mapsto x \vee j\bot$, while its co-pseudocomplement in the co-frame of sublocales is given by the nucleus $x \mapsto \bigwedge\{jy \to y \mid x \leq y\}$.[21]

We can now come back to our problem. Let X be the opposite of the ordinal $\omega+1$; more explicitly, X is obtained by taking the natural numbers $\mathbb{N}$, then reversing their natural order and, finally, adding a bottom element $\bot$ (the top element $\top$ is now 0). This is easily seen to be a frame. Here we consider the sublocale $X_{\neg\neg}$. Since it is dense, we have $cX_{\neg\neg} = X$; so $-cX_{\neg\neg}$ is the smallest sublocale of X. On the contrary, we claim that $-X_{\neg\neg} = X$ so that also $i-X_{\neg\neg} = X$. Indeed, we have $\neg n = \bot$ for every $n \in \mathbb{N}$.[22] Therefore $\neg\neg y \to y$ is $\top$ if $y = \bot$, while it is y otherwise. From this one can easily see that $\bigwedge\{\neg\neg y \to y \mid x \leq y\}$ is always equal to x.

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[21] In general, given two nuclei j and k, the map $x \mapsto \bigwedge\{jy \to ky \mid x \leq y\}$ defines a nucleus which turns to be the implication $j \to k$ in the frame of nuclei [9, page 52]. In particular, the nucleus $x \mapsto \bigwedge\{jy \to y \mid x \leq y\}$ is the pseudocomplement of j (because the identity map is the smallest nucleus) in the frame of nuclei, and hence it corresponds to the co-pseudocomplement of X_j in the co-frame of sublocales.

[22] This happens for every non-bottom element in every linearly ordered frame.

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