Journal of Logic & Analysis 18:4 (2026) 1–14
ISSN 1759-9008

On the isomorphism problem for ultraproducts of C*-algebras in continuous model theory

AKIHIKO ARAI

Abstract: In classical model theory, the Keisler–Shelah theorem establishes a fundamental connection between the elementary equivalence of structures and the isomorphism of their ultrapowers. Motivated by this, one may ask whether an analogous relationship holds in the framework of continuous model theory, which naturally encompasses metric structures such as C*-algebras. In this paper, we investigate the isomorphism problem for ultraproducts of operator algebras from a model-theoretic perspective. We prove that, assuming the negation of the continuum hypothesis, there exist two elementarily equivalent infinite-dimensional unital C*-algebras A and B, whose density characters are at most $\mathfrak{c}$, such that for all non-principal ultrafilters $\mathcal{U}, \mathcal{V}$ on ω, the ultrapowers $A^{\mathcal{U}}$ and $B^{\mathcal{V}}$ are not isomorphic. This result provides a continuous analogue of certain classical theorems concerning ultraproducts and demonstrates that the model-theoretic behavior of C*-algebras is closely related to set-theoretic principles such as the continuum hypothesis.

2020 Mathematics Subject Classification: 03C20, 46L05, 03C90, 03C98, 03E50 (primary)

Keywords: model theory, continuous logic, C*-algebras, ultraproduct, continuum hypothesis

1 Introduction

The ultraproduct construction is a significant object of study in both model theory and operator algebras. The model-theoretic notion of ultraproducts was introduced by Łoś in [22] in the 1950s, and Robinson made use of it to develop nonstandard analysis (cf. [24]). Around the same time, an ultraproduct-like construction appeared in the context of operator algebras in the work of Kaplansky and Wright [29]. Subsequently, Sakai introduced the ultraproduct of II_1 factors in [25], and McDuff [23] discovered its significance. Later, Connes [5] and Kirchberg [21] applied this construction to the classification theory of operator algebras. However, no essential connection between these two perspectives seems to have been recognized at the time.

Published: May 2026 https://doi.org/10.4115/jla.2026.18.4

In model theory, the isomorphism of ultraproducts of two structures is closely related to the notion of *elementary equivalence*. One of the earliest results in this direction is due to Keisler [20], who showed that, under the assumption of the continuum hypothesis, for any $\mathcal{L}$-theory T in a countable language $\mathcal{L}$ and any two models M and N of T of cardinality at most $\mathfrak{c}$, M and N are elementarily equivalent (i.e., $M \equiv N$) if and only if they have isomorphic ultrapowers with respect to an ultrafilter on ω. Furthermore, Shelah [26] (see also [27]) generalized this result by weakening the assumption on the ultrafilter from ω to sufficiently large cardinals, thereby showing that the theorem holds in ZFC. This is known as the *Keisler–Shelah theorem*. Thus, the following question naturally arises.

Question 1.1 *Assume the continuum hypothesis fails. For which $\mathcal{L}$-theories T in a countable language $\mathcal{L}$, do there exist models M and N of T of size at most $\mathfrak{c}$ (or $\aleph_0$) such that M and N are elementarily equivalent, but $M^{\mathcal{U}}$ and $N^{\mathcal{V}}$ are not isomorphic for all non-principal ultrafilters $\mathcal{U}, \mathcal{V}$ on ω?*

Several concrete results, as well as sufficient conditions for Question 1.1, have been obtained by Golshani and Shelah [15, 16], Tsuboi [28], and Goto [17].

In recent years, *continuous model theory*, a generalization of classical model theory to the setting of metric structures, has gained increasing attention as a powerful tool for analyzing ultraproducts of operator algebras. For example, Farah, Hart, and Sherman showed in this framework that all countable ultrapowers of a separable C^*-algebra A are isomorphic if and only if the continuum hypothesis holds [9, Theorem 5.1].

Accordingly, we are interested in the following problem, which is an extension of Question 1.1 to the framework of continuous model theory, particularly in the case of the theory of operator algebras. (It is mentioned by Farah, Hart, and Sherman in [11, § 4] in the case of II_1 factors.)

Question 1.2 *Assume the continuum hypothesis fails. For which $\mathcal{L}$-theories T with $\chi(\mathrm{T}, \mathcal{L}) \leq \aleph_0$ do there exist models M and N of T of density character at most $\mathfrak{c}$ (or $\aleph_0$) such that $M \equiv N$, but $M^{\mathcal{U}} \not\cong N^{\mathcal{V}}$ for all non-principal ultrafilters $\mathcal{U}, \mathcal{V}$ on ω?*

In this paper, we study this problem in the setting of C^*-algebras. For Question 1.2, we obtain the following result:

Theorem 1.3 *Assume the continuum hypothesis fails. Then there exist C^*-algebras A and B of density character at most $\mathfrak{c}$ such that $A \equiv B$, but $A^{\mathcal{U}}$ and $B^{\mathcal{V}}$ are not isomorphic for any non-principal ultrafilters $\mathcal{U}, \mathcal{V}$ on ω.*

We describe the overall outline of this paper. In Section 2, we present preliminaries, including several basic results in continuous model theory.

In Section 3, we provide a sufficient condition for Question 1.1 via Theorem 3.1. This result applies to general continuous languages, and in the classical setting it yields several known results; see Remark 3.3.

In Section 4, we prove Theorem 1.3 based on the results of Section 3. In particular, Lemma 4.1 provides a concrete example of a theory satisfying Theorem 1.3.

In Section 5, we discuss a topological application of the results obtained in Section 4. Finally, in Section 6, we propose several related open problems as directions for future research.

Acknowledgement

I am grateful to the anonymous referee, whose numerous comments and suggestions helped to improve the exposition and sharpen the results of the paper substantially. I would like to express my deepest gratitude to my supervisor, Hiroki Matui, for his continuous guidance and encouragement. I am also grateful to Kota Takeuchi and all members of his laboratory for their helpful comments and discussions through seminars. Finally, I would like to thank Tatsuya Goto, who introduced this problem at the Model Theory Summer School and Workshop 2024.

2 Preliminaries

In this section, we present several basic results in continuous model theory, without proofs. For detailed proofs and further discussion, we refer the reader to Hart [18], Farah, Hart, and Sherman [10], Farah et al. [8], and Ben Yaacov, Berenstein, Henson, and Usvyatsov [1].

Notation 2.1

- *We denote ordinals by* $\alpha, \beta, \gamma, \dots$ *and cardinals by* $\kappa, \lambda, \dots$.
- *For a cardinal* κ, *we denote by* κ^+ *the successor cardinal of* κ, *that is, the least cardinal greater than* κ.
- *We denote by* ω_α *the initial ordinal of cardinality* $\aleph_\alpha$. *In particular,* ω_0, *which is the set of all natural numbers, is simply denoted by* ω.

- *We denote the cardinality of the continuum* $2^{\aleph_0}$ *by* $\mathfrak{c}$. *The* continuum hypothesis *(CH) is the statement* $\mathfrak{c} = \aleph_1$. *Thus,* $\neg$*CH is equivalent to* $\aleph_1 < \mathfrak{c}$, *and hence to* $\aleph_2 \leq \mathfrak{c}$.
- *For a set* X *equipped with a binary relation* R, *the* cofinality *of* (X, R), *denoted by* $\mathrm{cf}(X, R)$, *is the least cardinality of a cofinal subset of* (X, R).
- *We denote by* $\beta I \setminus I$ *the set of all non-principal ultrafilters on a set* I.

We use the notation of continuous model theory introduced by Hart in [18]. From now on, unless otherwise specified, we assume that $\mathcal{L}$ is a language (in the sense of continuous model theory), and that T is a complete theory with $\chi(\mathrm{T}, \mathcal{L}) \leq \aleph_0$.

Many fundamental results in classical model theory admit natural analogues in continuous model theory. For the reader's convenience, we list below several relevant theorems together with references:

- Łoś' theorem (see Ben Yaacov, Berenstein, Henson, and Usvyatsov [1, Theorem 5.4] for a complete proof),
- The downward Löwenheim–Skolem–Tarski theorem (see [18, Theorem 5.6] and [10, Theorem 4.6]),
- The Tarski's elementary chain theorem (cf. [18, Proposition 5.10]).
- A characterization of definable sets (see [18, Theorem 8.2]).
- The existence of saturated models (see [18, Proposition 7.8]).

Finally, we conclude this section with some remarks.

Remark 2.2 *Let* $(M_\alpha)_{\alpha<\gamma}$ *be an elementary chain of* $\mathcal{L}$*-structures, and let* $M = \bigcup_{\alpha<\gamma} M_\alpha$. *The continuous analogue of Tarski's elementary chain theorem states that the completion of* M *is an elementary extension of* M_α *for all* $\alpha < \gamma$. *However, if* $\mathrm{cf}(\gamma) > \aleph_0$, *then* M *is automatically complete. Therefore,* M *itself is an elementary extension of* M_α *for all* $\alpha < \gamma$.

The following result is not stated explicitly in [18], so we include its statement and proof here.

Lemma 2.3 *Let* M *be an* $\mathcal{L}$*-structure and let* $A \subseteq M$. *For a set* $t(\bar{x})$ *of* $\mathcal{L}_A$*-formulas with free variables* $\bar{x}$, *the following are equivalent:*

(1) $t(\bar{x})$ *is (the kernel of) a type over* A; *that is, there exist a model* $N \succeq M$ *and a tuple* $\bar{b} \in N$ *such that* $\varphi^N(\bar{b}) = 0$ *for all* $\varphi(\bar{x}) \in t(\bar{x})$.

(2) *$t(\bar{x})$ is approximately finitely satisfiable in M. i.e., for every finite subset $t_0(\bar{x}) \subseteq t(\bar{x})$ and every $\varepsilon > 0$, there exists $\bar{b} \in M$ such that $\left|\varphi^M(\bar{b})\right| < \varepsilon$ for all $\varphi(\bar{x}) \in t_0(\bar{x})$.*

Proof The implication (2) $\implies$ (1) follows immediately from the compactness theorem for continuous model theory (see [18, Theorem 6.4]).

We show (1) $\implies$ (2). By (1), there exists a model $N \succeq M$ such that $t(\bar{x})$ is realized in N. Fix a finite subset $t_0(\bar{x}) \subseteq t(\bar{x})$. The $\mathcal{L}_A$-sentence

$$\inf_{\bar{x}} \max \left\{ |\varphi(\bar{x})| : \varphi(\bar{x}) \in t_0(\bar{x}) \right\}$$

belongs to $\mathrm{Th}(N) = \mathrm{Th}(M_A)$. Thus, for every $\varepsilon > 0$, there exists some $\bar{b} \in M$ such that $\left|\varphi^M(\bar{b})\right| < \varepsilon$ holds for all $\varphi(\bar{x})$ in $t_0(\bar{x})$, which completes the proof of (2). □

3 Main Result for General Setting

In this section, we give a sufficient condition on T for Theorem 1.3 in the general setting of continuous model theory.

Theorem 3.1 *Assume that T has a definable set X relative to T and a (possibly definable) formula $\varphi(\bar{x}, \bar{y})$ satisfying the following conditions:*

- *φ takes discrete values $\{0, 1\}$ on X. That is, for every $M \models \mathrm{T}$, we have $\varphi^M(\bar{a}, \bar{b}) \in \{0, 1\}$ for all $\bar{a}, \bar{b} \in X(M)$.*
- *φ defines an asymmetric relation on X. That is, for every $M \models \mathrm{T}$ and all $\bar{a}, \bar{b} \in X(M)$, if $\varphi^M(\bar{a}, \bar{b}) = 0$, then $\varphi^M(\bar{b}, \bar{a}) = 1$.*
- *For any $M \models \mathrm{T}$, any $n < \omega$, and any $\bar{a}_i \in X(M)$ $(i < n)$, there exists $\bar{b} \in X(M)$ such that $\varphi^M(\bar{a}_i, \bar{b}) = 0$ for all $i < n$.*

Then, under the failure of the continuum hypothesis, there exist models $A, B \models \mathrm{T}$ with density characters at most $\mathfrak{c}$ such that $A^{\mathcal{U}} \not\cong B^{\mathcal{V}}$ for all $\mathcal{U}, \mathcal{V} \in \beta\omega \setminus \omega$.

Proof We take a κ-saturated model M of T for sufficiently large cardinals κ.

For $\bar{a}, \bar{b} \in X(M)$, define $\bar{a} \lhd \bar{b}$ by $\varphi^M(\bar{a}, \bar{b}) = 0$. Let $\bar{a} \unlhd \bar{b}$ denote $\bar{a} \lhd \bar{b}$ or $\bar{a} = \bar{b}$.

For ordinals $\alpha \leq \mathfrak{c}$, we define $\bar{a}_\alpha \in M$ and $M_\alpha \preceq M$ inductively that satisfy the following conditions:

- For all $\alpha < \mathfrak{c}$, we have $M_\alpha \preceq M_{\alpha+1}$ and $\chi(M_\alpha) \leq |\alpha| + \aleph_0$.
- For all $\alpha < \mathfrak{c}$, we have $\bar{a}_\alpha \in X(M_{\alpha+1})$ and $\bar{a} \lhd \bar{a}_\alpha$ for all $\bar{a} \in X(M_\alpha)$.
- For every limit ordinal $\gamma \leq \mathfrak{c}$, we have $M_\gamma = \overline{\bigcup_{\alpha<\gamma} M_\alpha}$.

One can find a separable elementary submodel M_0 of M by the downward Löwenheim–Skolem–Tarski theorem. Suppose that M_α has already been defined. Consider the set of formulas

$$t(\bar{x}) := \{d(\bar{x}, X)\} \cup \{\varphi(\bar{a}, \bar{x}) : \bar{a} \in X(M_\alpha)\}.$$

By the assumptions on φ, $t(\bar{x})$ is approximately finitely satisfiable. Thus, by Lemma 2.3, $t(\bar{x})$ is a type over M. By the saturation of M, we can choose $\bar{a}_\alpha \in M$ realizing $t(\bar{x})$. By the downward Löwenheim–Skolem–Tarski theorem, we can take $M_{\alpha+1} \preceq M$ such that $M_\alpha \cup \{\bar{a}_\alpha\} \subseteq M_{\alpha+1}$ and $\chi(M_{\alpha+1}) \leq \chi(M_\alpha) + \aleph_0$. For a limit ordinal $\gamma \leq \mathfrak{c}$, we note that $M_\gamma \preceq M$ by the Tarski's elementary chain theorem.

In this setting, the following holds:

Claim *For every uncountable regular cardinal $\kappa \leq \mathfrak{c}$ and every $\mathcal{U} \in \beta\omega \setminus \omega$, we have*

$$\operatorname{cf}\left(X(M_\kappa^{\mathcal{U}}), \unlhd\right) = \kappa.$$

Proof of claim. First, we show that $\operatorname{cf}(X(M_\kappa^{\mathcal{U}}), \unlhd) \leq \kappa$. Let

$$C := \{[(\bar{a}_\alpha)_n] : \alpha < \kappa\} \subseteq X(M_\kappa^{\mathcal{U}}).$$

Obviously, $|C| \leq \kappa$. We show that C is cofinal in $(X(M_\kappa^{\mathcal{U}}), \unlhd)$. Let $\bar{d} \in X(M_\kappa^{\mathcal{U}})$. Since definable sets commute with ultraproducts, we may assume that $\bar{d} = [(\bar{d}_n)_n]_{\mathcal{U}}$, where $\bar{d}_n \in X(M_\kappa)$ for all $n < \omega$. By the regularity of κ and Remark 2.2, for each $n < \omega$ we may choose $\alpha_n < \kappa$ such that $\bar{d}_n \in X(M_{\alpha_n})$. Let $\alpha := \sup_{n<\omega} \alpha_n$ and define $\bar{c} := [(\bar{a}_\alpha)_{n<\omega}]_{\mathcal{U}} \in C$. Since κ is a regular cardinal, we have $\alpha < \kappa$, and hence $\bar{a}_\alpha \in X(M_\kappa)$. For all $n < \omega$, we have $\bar{d}_n \unlhd \bar{a}_\alpha$ Therefore, by the Łoś' theorem, we obtain $\bar{d} \unlhd \bar{c}$.

Next, we show that $\kappa \leq \operatorname{cf}(X(M_\kappa^{\mathcal{U}}), \unlhd)$. Let $C \subseteq X(M_\kappa^{\mathcal{U}})$ be a subset with $|C| < \kappa$. Write $C = \{\bar{c}^\alpha : \alpha < \lambda\}$, where $\lambda := |C| < \kappa$, and $\bar{c}^\alpha = [(\bar{c}_n^\alpha)_n]_{\mathcal{U}}$ with $\bar{c}_n^\alpha \in X(M_\kappa)$ for each $n < \omega$ and $\alpha < \lambda$. By the regularity of κ and Remark 2.2, for each $n < \omega$ and $\alpha < \lambda$, we can take $\beta_n^\alpha < \kappa$ such that $\bar{c}_n^\alpha \in M_{\beta_n^\alpha}$. Let $\beta := \sup\{\beta_n^\alpha : n < \omega, \alpha < \lambda\}$ and $\bar{d} := [(\bar{a}_\beta)_n]_{\mathcal{U}}$. Since κ is regular and $\lambda < \kappa$, we have $\beta < \kappa$, and hence $\bar{a}_\beta \in X(M_\kappa)$. For each $\alpha < \lambda$ and every $n < \omega$, we have $\bar{c}_n^\alpha \lhd \bar{a}_\beta$, and hence $\bar{c}^\alpha \lhd \bar{d}$ by the Łoś' theorem. Since $\lhd$ is an asymmetric relation, it follows that $\bar{d} \ntrianglelefteq \bar{c}^\alpha$ for all $\alpha < \lambda$. $\square$

Assume the continuum hypothesis fails. Then we have $\omega_2 \leq \mathfrak{c}$. Thus, by the claim above, letting $A := M_{\omega_1}$ and $B := M_{\omega_2}$, we obtain

$$\mathrm{cf}(X(A^{\mathcal{U}}), \trianglelefteq) = \omega_1 \neq \omega_2 = \mathrm{cf}(X(B^{\mathcal{V}}), \trianglelefteq)$$

for any $\mathcal{U}, \mathcal{V} \in \beta\omega \setminus \omega$, thereby completing the proof. □

The above result can be regarded as a generalization of [15, Theorem 2.1] and the results obtained in [28, § 3].

Remark 3.2 *The conditions of Theorem 3.1 can be described in terms of formulas as follows:*

- $\mathrm{T} \models \sup_{\bar{x},\bar{y}\in X} \min\{|\varphi(\bar{x},\bar{y})|, |1-\varphi(\bar{x},\bar{y})|\}$.
- $\mathrm{T} \models \sup_{\bar{x},\bar{y}\in X} \max\{0, 1-\varphi(\bar{y},\bar{x})-\varphi(\bar{x},\bar{y})\}$.
- *For all* $n < \omega$, *we have*

$$\mathrm{T} \models \sup_{\bar{x}_0\in X} \dots \sup_{\bar{x}_{n-1}\in X} \inf_{\bar{y}\in X} \max_{i<n} \varphi(\bar{x}_i, \bar{y}).$$

Consequently, since T *is assumed to be complete, if these conditions hold in some model of* T, *then they hold in every model of* T.

Remark 3.3 *In the classical setting, condition (1) of Theorem 3.1 is not needed, so conditions (2) and (3) suffice. In this case, several known results follow from Theorem 3.1:*

- $\mathrm{Th}(\mathbb{Q}, <)$ *([15, Theorem 2.1]), with* $\varphi(x,y) \equiv x < y$.
  - *More generally, any (strictly) directed set satisfies the conditions of Theorem 3.1.*
- *The theory of random graphs ([28]) :* $\varphi((x,y),(u,v)) \equiv \neg(xRv) \wedge yRu$.
- *The theory of atomless Boolean algebras ([17]) :*

$$\varphi((a,b),(a',b')) \equiv (ab' \neq 0 \wedge a(1-b') \neq 0) \wedge (a'b = 0 \vee a'(1-b) = 0).$$

Remark 3.4 *It is known that Goldbring and Keisler proved that the Keisler–Shelah theorem also holds in the continuous setting [14]. Therefore, the result of Theorem 3.1 indicates that the failure of the continuum hypothesis affects the cardinalities of the index sets of ultrapowers.*

4 Proof of Theorem 1.3

In order to prove Theorem 1.3, it suffices to find a theory of unital C^*-algebras that satisfies the conditions of Theorem 3.1. For a C^*-algebra A, $\mathcal{P}(A)$ denotes the set of all projections in A. For $p \in \mathcal{P}(A) \setminus \{0\}$, p is said to be *minimal* if, for any $q \in \mathcal{P}(A)$ with $0 \leq q \leq p$ one has $q = 0$ or $q = p$.

Lemma 4.1 *Let* T *be a theory of unital* C^**-algebras* A *satisfying the following conditions:*

- *For all* $p, q \in \mathcal{P}(A)$*, we have* $pq = qp$*.*
- A *has no minimal projections. That is, for every* $p \in \mathcal{P}(A) \setminus \{0\}$*, there exists* $q \in \mathcal{P}(A)$ *such that* $0 \lneq q \lneq p$*.*

Then, there exists a formula that satisfies the conditions in Theorem 3.1.

Proof Let $X(\cdot) := \mathcal{P}(\cdot) \setminus \{0,1\}$. We note that X is a definable set relative to the theory of unital C^*-algebras (cf. [18, Example 8.5]).

Define the formulas $\varphi(x,y)$ and $\psi((x,y),(x',y'))$ by

$$\begin{aligned} \varphi(x,y) &:= \max\{1 - \|xy\|, 1 - \|x(1-y)\|\}, \\ \psi((x,y),(x',y')) &:= \max\{\varphi(x,y'), 1 - \varphi(x',y)\}. \end{aligned}$$

By the commutativity of projections, both pq and $p(1-q)$ are projections for all $p, q \in X(A)$. Hence, φ and ψ take only the discrete values $\{0,1\}$ on X. Moreover, it is easy to check that ψ is asymmetric on X^2.

To show that ψ satisfies condition (3) of Theorem 3.1, it suffices to prove the following claim:

Claim *For every* $n < \omega$ *and* $p_i \in X(A)$ $(i < n)$*, there exist* $q, r \in X(A)$ *such that*

$$\varphi^A(p_i, q) = 0 \quad \text{and} \quad \varphi^A(r, p_i) = 1$$

for all $i < n$*.*

We now prove this claim. Let $n < \omega$ and let $p_i \in X(A)$ for $i < n$.

For each $\sigma \in 2^n$, we define $p_\sigma \in X(A)$ by

$$p_\sigma := \prod_{i<n} [(1 - \sigma(i))p_i + \sigma(i)(1 - p_i)].$$

Since A has no minimal projections, for each $\sigma \in 2^n$ with $p_\sigma \neq 0$, we can choose $q_\sigma \in X(A)$ such that $0 \lneq q_\sigma \lneq p_\sigma$, and define $q := \sum_\sigma q_\sigma$. Since the sum of all p_σ is equal to the unit 1, there exists $\sigma \in 2^n$ such that $p_\sigma \neq 0$, and thus $q \neq 0$. Moreover, since the p_σ are mutually orthogonal, the q_σ are also mutually orthogonal, and hence $q \in X(A)$. For each $i < n$, we have $p_i q = \sum_{\sigma:\sigma(i)=0} q_\sigma \neq 0$ and $p_i(1-q) = \sum_{\sigma:\sigma(i)=0}(p_\sigma - q_\sigma) \neq 0$. Therefore, $\varphi^A(p_i, q) = 0$ for all $i < n$.

Next, define $r := p_\sigma \in X(A)$ for some $\sigma \in 2^n$ with $p_\sigma \neq 0$. For each $i < n$, if $\sigma(i) = 0$, then $r = p_\sigma \leq p_i$, and hence $r(1-p_i) = 0$. Similarly, if $\sigma(i) = 1$, then $rp_i = 0$. Therefore, $\varphi^A(r, p_i) = 1$ for all $i < n$. □

The following is our main result.

Theorem 4.2 *Suppose that* T *satisfies the conditions in Lemma 4.1. Then, under the failure of the continuum hypothesis, there exist unital* C* *-algebras* $A, B \models \mathrm{T}$ *with density characters at most* $\mathfrak{c}$ *such that* $A^{\mathcal{U}} \ncong B^{\mathcal{V}}$ *for all* $\mathcal{U}, \mathcal{V} \in \beta\omega \setminus \omega$.

Proof This immediately follows from Theorem 3.1 and Lemma 4.1. □

Finally, we conclude this section by giving a concrete example of Theorem 1.3.

Example 4.3 *Let* X *be the Cantor space* 2^ω. *The* C* *-algebra* $A = C(X)$ *is an abelian unital* C* *-algebra with no minimal projections. Thus, the theory of* A *satisfies the conditions in Lemma 4.1. More generally, for any abelian unital* C* *-algebra* $A = C(X)$, *where* X *is a compact Hausdorff space,* A *has no minimal projections if and only if the Boolean algebra* $\mathrm{clop}(X)$ *consisting of all clopen subsets of* X *is atomless.*

A nonabelian example is obtained by tensoring a unital nonabelian C* *-algebra with no nontrivial projections such as the Jiang–Su algebra* $\mathcal{Z}$. *Namely, for* X *as above, the theory of* $\mathcal{Z} \otimes C(X)$ *satisfies the conditions in Lemma 4.1.*

5 Applications to General Topology

In this section, we discuss applications of Theorem 1.3 to general topology in the commutative case.

Definition 5.1 *Suppose that X is a topological space. We say that X is* totally disconnected *if all connected components of X are singleton sets. We say that X is* 0-dimensional *if $\mathrm{clop}(X)$ forms a basis for X. When X is a compact Hausdorff space, the conditions of being totally disconnected and being 0-dimensional are equivalent. A compact Hausdorff space satisfying these conditions is called a* Stone space.

We note that Gelfand duality implies that, for compact Hausdorff spaces X and Y, X and Y are homeomorphic if and only if $C(X) \cong C(Y)$ as C*-algebras. Similarly, Stone duality, as discussed by Farah [7, § 1.3.1], implies that, for Stone spaces X and Y, X and Y are homeomorphic if and only if $\mathrm{clop}(X) \cong \mathrm{clop}(Y)$ as Boolean algebras. Moreover, a compact Hausdorff space X is a Stone space if and only if $C(X)$ is an abelian C*-algebra of real rank zero.

Lemma 5.2 (Eagle and Vignati [6, Lemma 5.7]) *Let X be a compact Hausdorff space, and $\mathcal{U}$ be an ultrafilter. Then $C(X)^{\mathcal{U}} \cong C(\sum_{\mathcal{U}} X)$, and $\mathrm{clop}(X)^{\mathcal{U}} \cong \mathrm{clop}(\sum_{\mathcal{U}} X)$ where $\sum_{\mathcal{U}} X$ is the ultracopower of X and $\mathrm{clop}(X)^{\mathcal{U}}$ is the classical model-theoretic ultrapower of Boolean algebra $\mathrm{clop}(X)$.*

For a topological space X, the *weight* of X, denoted by $w(X)$, is the least cardinality of a basis for X.

By applying Theorem 4.2 to $C(2^\omega)$, we obtain the following result:

Corollary 5.3 *Under the failure of the continuum hypothesis, there exist Stone spaces X and Y satisfying the following properties:*

(1) $w(X), w(Y) \leq \mathfrak{c}$.

(2) *There exist a set I and non-principal ultrafilters $\mathcal{U}, \mathcal{V} \in \beta I \setminus I$ such that $\sum_{\mathcal{U}} X$ and $\sum_{\mathcal{V}} Y$ are homeomorphic.*

(3) *For any $\mathcal{U}, \mathcal{V} \in \beta\omega \setminus \omega$, $\sum_{\mathcal{U}} X$ and $\sum_{\mathcal{V}} Y$ are not homeomorphic.*

Proof Suppose that the continuum hypothesis fails. By Theorem 4.2 with $M = C(2^\omega)$, there exist $A, B \models \mathrm{Th}(M)$ such that $\chi(A), \chi(B) \leq \mathfrak{c}$ and $A^{\mathcal{U}} \not\cong B^{\mathcal{V}}$ for all $\mathcal{U}, \mathcal{V} \in \beta\omega \setminus \omega$. By the Keisler–Shelah theorem, there exist a set I and non-principal ultrafilters $\mathcal{U}, \mathcal{V} \in \beta I \setminus I$ such that $A^{\mathcal{U}} \cong B^{\mathcal{V}}$.

Note that M is a unital abelian C*-algebra of real rank zero. Since the class of real rank zero C*-algebras is elementary (see [8, Example 2.4.2]), A and B are also unital abelian C*-algebras of real rank zero. Thus, there exist Stone spaces X and Y such that $A \cong C(X)$ and $B \cong C(Y)$.

Since, for every infinite Hausdorff space Z, the density character of $C(Z)$ coincides with $w(Z)$ (see [12, Lemma 1.2]), we have $w(X), w(Y) \leq \mathfrak{c}$; that is, condition (1) holds. Moreover, conditions (2) and (3) also hold for X and Y, by the properties of A and B and by Lemma 5.2. $\square$

6 Conclusion

In conclusion, we discuss generalizations of Theorem 3.1 and Theorem 4.2. We note that if a theory T is *stable* (in the sense of continuous model theory; cf. [10, Definition 5.2]), then all ultrapowers of models of T with density character at most $\mathfrak{c}$ are isomorphic without assuming the continuum hypothesis, since they are $\mathfrak{c}$-saturated (cf. [10, Theorem 5.6 (1)]). Thus, we are led to consider the following question:

Question 6.1 *Assume the continuum hypothesis fails. Let* T *be an unstable* $\mathcal{L}$*-theory. Do there exist* $A, B \models \mathrm{T}$ *with* $\chi(A), \chi(B) \leq \mathfrak{c}$ *such that* $A^{\mathcal{U}} \ncong B^{\mathcal{V}}$ *for all* $\mathcal{U}, \mathcal{V} \in \beta\omega \backslash \omega$*?*

We note that the assumptions of Theorem 3.1 immediately imply that the theory T has the *order property* (cf. [10, Definition 5.2]), which is equivalent to T being unstable (see [10, Theorem 5.5]). It is known that for any unital infinite-dimensional C^*-algebra A, the theory $\mathrm{Th}(A)$ has the order property (see [9, Lemma 5.3]). Therefore, as a special case of the above question, the following question is also of interest.

Question 6.2 *Assume the continuum hypothesis fails. Let* T *be the theory of a unital infinite-dimensional* C^**-algebra. Do there exist* $A, B \models \mathrm{T}$ *with* $\chi(A), \chi(B) \leq \mathfrak{c}$ *such that* $A^{\mathcal{U}} \ncong B^{\mathcal{V}}$ *for all* $\mathcal{U}, \mathcal{V} \in \beta\omega \setminus \omega$*?*

In the construction given in the proof of Theorem 4.2, the density characters of A and B are controlled only up to $\leq \aleph_1$ and $\leq \aleph_2$, respectively. Therefore, the case where A and B are separable is still unsolved.

Question 6.3 *Assume that the continuum hypothesis fails. Does there exist a theory* T *of* C^**-algebras with separable models* $A, B \models \mathrm{T}$ *such that* $A^{\mathcal{U}} \ncong B^{\mathcal{V}}$ *for all* $\mathcal{U}, \mathcal{V} \in \beta\omega \setminus \omega$*?*

In the classical setting of Question 6.3, Shelah [27] gave a result for the theory of graphs. By the discussion of Corollary 5.3, the abelian C^*-algebra case of Question 6.3 can be reformulated as the following question:

Question 6.4 *Assume the continuum hypothesis fails. Do there exist Stone spaces X and Y satisfying the following properties?*

(1) $w(X), w(Y) \leq \aleph_0$, *that is,* X *and* Y *are second countable.*

(2) *There exist a set* I *and non-principal ultrafilters* $\mathcal{U}, \mathcal{V} \in \beta I \setminus I$ *such that* $\sum_{\mathcal{U}} X$ *and* $\sum_{\mathcal{V}} Y$ *are homeomorphic.*

(3) *For any* $\mathcal{U}, \mathcal{V} \in \beta\omega \setminus \omega$, $\sum_{\mathcal{U}} X$ *and* $\sum_{\mathcal{V}} Y$ *are not homeomorphic.*

Remark 6.5 *By the Riesz–Markov–Kakutani theorem, for a compact Hausdorff space* X, $C(X)$ *is separable if and only if* X *is metrizable. Hence, in Question 6.4 the requirement that* X *and* Y *are second countable can be replaced with the requirement that* X *and* Y *are metrizable.*

References

[1] **I Ben Yaacov**, **A Berenstein**, **C W Henson**, **A Usvyatsov**, *Model theory for metric structures*, from: "Model theory with applications to algebra and analysis. Vol. 2", London Math. Soc. Lecture Note Ser. 350, Cambridge Univ. Press, Cambridge (2008) 315–427; https://doi.org/10.1017/CBO9780511735219.011

[2] **B Blackadar**, *Operator algebras. Theory of* C^**-algebras and von Neumann algebras*, volume 122 of *Encyclopaedia of Mathematical Sciences*, Springer-Verlag, Berlin (2006)Operator Algebras and Non-commutative Geometry, III. ; https://doi.org/10.1007/3-540-28517-2

[3] **C C Chang**, **H J Keisler**, *Continuous model theory*, volume 58 of *Annals of Mathematics Studies*, Princeton University Press, Princeton, NJ (1966); https://doi.org/10.1515/9781400882052

[4] **C C Chang**, **H J Keisler**, *Model theory*, volume 73 of *Studies in Logic and the Foundations of Mathematics*, third edition, North-Holland Publishing Co., Amsterdam (1990)

[5] **A Connes**, *Classification of injective factors. Cases* $\mathrm{II}_1, \mathrm{II}_\infty, \mathrm{III}_\lambda, \lambda \neq 1$, Ann. of Math. (2) 104 (1976) 73–115; https://doi.org/10.2307/1971057

[6] **C J Eagle**, **A Vignati**, *Saturation and elementary equivalence of* C^**-algebras*, Journal of Functional Analysis 269 (2015) 2631–2664; https://doi.org/10.1016/j.jfa.2015.04.013

[7] **I Farah**, *Combinatorial set theory of* C^**-algebras*, Springer Monographs in Mathematics, Springer, Cham ([2019] ©2019); https://doi.org/10.1007/978-3-030-27093-3

[8] **I Farah**, **B Hart**, **M Lupini**, **L Robert**, **A Tikuisis**, **A Vignati**, **W Winter**, *Model theory of* C^**-algebras*, Mem. Amer. Math. Soc. 271 (2021) viii+127; https://doi.org/10.1090/memo/1324

[9] **I Farah**, **B Hart**, **D Sherman**, *Model theory of operator algebras I: stability*, Bull. Lond. Math. Soc. 45 (2013) 825–838; https://doi.org/10.1112/blms/bdt014

[10] **I Farah**, **B Hart**, **D Sherman**, *Model theory of operator algebras II: model theory*, Israel J. Math. 201 (2014) 477–505; https://doi.org/10.1007/s11856-014-1046-7

[11] **I Farah**, **B Hart**, **D Sherman**, *Model theory of operator algebras III: elementary equivalence and* II_1 *factors*, Bull. Lond. Math. Soc. 46 (2014) 609–628; https://doi.org/10.1112/blms/bdu012

[12] **I Farah**, **M Magidor**, *Corson reflections*, Ann. Pure Appl. Logic 172 (2021) Paper No. 102908, 13; https://doi.org/10.1016/j.apal.2020.102908

[13] **L Ge**, **D Hadwin**, *Ultraproducts of C^* -algebras*, from: "Recent advances in operator theory and related topics (Szeged, 1999)", Oper. Theory Adv. Appl. 127, Birkhäuser, Basel (2001) 305–326; https://doi.org/10.1007/978-3-0348-8374-0_17

[14] **I Goldbring**, **H J Keisler**, *Continuous sentences preserved under reduced products*, J. Symb. Log. 87 (2022) 649–681; https://doi.org/10.1017/jsl.2020.28

[15] **M Golshani**, **S Shelah**, *The Keisler-Shelah isomorphism theorem and the continuum hypothesis*, Fund. Math. 260 (2023) 59–66; https://doi.org/10.4064/fm198-5-2022

[16] **M Golshani**, **S Shelah**, *The Keisler-Shelah isomorphism theorem and the continuum hypothesis II*, Monatsh. Math. 201 (2023) 789–801; https://doi.org/10.1007/s00605-023-01837-y

[17] **T Goto**, *Failure of Keisler-Shelah theorem for atomless Boolean algebras*, A talk given in the model theory summer school & workshop, Kobe University, Japan (2024)

[18] **B Hart**, *An introduction to continuous model theory*, from: "Model Theory of Operator Algebras", (I Goldbring, editor), De Gruyter, Berlin, Boston (2023) 83–132; https://doi.org/doi:10.1515/9783110768282-003

[19] **X Jiang**, **H Su**, *On a simple unital projectionless C^* -algebra*, Amer. J. Math. 121 (1999) 359–413; https://doi.org/10.1353/ajm.1999.0012

[20] **H J Keisler**, *Ultraproducts and elementary classes*, Indag. Math. 23 (1961) 477–495Nederl. Akad. Wetensch. Proc. Ser. A **64**. ; https://doi.org/10.1016/s1385-7258(61)50048-0

[21] **E Kirchberg**, *Exact* C* *-algebras, tensor products, and the classification of purely infinite algebras*, from: "Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Zürich, 1994)", Birkhäuser, Basel (1995) 943–954; https://doi.org/10.1007/978-3-0348-9078-6_87

[22] **J Łoś**, *Quelques remarques, théorèmes et problèmes sur les classes définissables d'algèbres*, from: "Mathematical interpretation of formal systems", North-Holland, Amsterdam (1955) 98–113; https://doi.org/10.1016/s0049-237x(09)70306-4

[23] **D McDuff**, *Central sequences and the hyperfinite factor*, Proc. London Math. Soc. (3) 21 (1970) 443–461; https://doi.org/10.1112/plms/s3-21.3.443

- [24] **A Robinson**, *Non-standard analysis*, North-Holland Publishing Co., Amsterdam (1966); https://doi.org/10.1515/9781400884223
- [25] **S Sakai**, *The Theory of W^*-algebras*, Lecture Notes, Yale University (1962)
- [26] **S Shelah**, *Every two elementarily equivalent models have isomorphic ultrapowers*, Israel J. Math. 10 (1971) 224–233; https://doi.org/10.1007/BF02771574
- [27] **S Shelah**, *Vive la différence. I. Nonisomorphism of ultrapowers of countable models*, from: "Set theory of the continuum (Berkeley, CA, 1989)", Math. Sci. Res. Inst. Publ. 26, Springer, New York (1992) 357–405; https://doi.org/10.1007/978-1-4613-9754-0_20
- [28] **A Tsuboi**, *Some results related to Keisler-Shelah isomorphism theorem*, from: "RIMS Kokyuroku", (H Kikyo, editor), volume 2218, Kyoto University Research Institute for Mathematical Sciences, Kyoto, Japan (2022) 35–39
- [29] **F B Wright**, *A reduction for algebras of finite type*, Ann. of Math. (2) 60 (1954) 560—570; https://doi.org/10.2307/1969851

Graduate School of Science and Engineering, Chiba University
1-33 Yayoicho, Inage-ku, Chiba-shi, Chiba, 263-8522 Japan

`23wd0105@student.gs.chiba-u.jp`

Received: 6 February 2026 Revised: 25 May 2026